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VARIATIONAL METHODS FOR THE SOLUTION OF PROBLEMS OF EQUILIBRIUM AND VIBRATIONS

R. COURANT

As Henri Poincaré once remarked, "solution of a mathematical problem" is a phrase of indefinite meaning. Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal. Such one sided views seem to reflect human limitations rather than objective values. In itself mathematics is an indivisible organism uniting theoretical contemplation and active application.

This address will deal with a topic in which such a synthesis of theoretical and applied mathematics has become particularly convincing. Since Gauss and W. Thompson, the equivalence between boundary value problems of partial differential equations on the one hand and problems of the calculus of variations on the other hand has been a central point in analysis. At first, the theoretical interest in existence proofs dominated and only much later were practical applications envisaged by two physicists, Lord Rayleigh and Walther Ritz; they independently conceived the idea of utilizing this equivalence for numerical calculation of the solutions, by substituting for the variational problems simpler approximating extremum problems in which but a finite number of parameters need be determined. Rayleigh, in his classical work—*Theory of sound*—and in other publications, was the first to use such a procedure. But only the spectacular success of Walther Ritz and its tragic circumstances caught the general interest. In two publications of 1908 and 1909 [39], Ritz, conscious of his imminent death from consumption, gave a masterly account of the theory, and at the same time applied his method to the calculation of the nodal lines of vibrating plates, a problem of classical physics that previously had not been satisfactorily treated.

Thus methods emerged which could not fail to attract engineers and physicists; after all, the minimum principles of mechanics are more suggestive than the differential equations. Great successes in applications were soon followed by further progress in the understanding of the theoretical background, and such progress in turn must result in advantages for the applications.

An address delivered before the meeting of the Society in Washington, D.C., on May 3, 1941, by invitation of the Program Committee; received by the editors June 16, 1942.

The following address will try to convey an idea of this interplay between practical and theoretical points of view. Completeness cannot be attempted; rather some selected topics with which the speaker has been personally concerned will be discussed.

Usually the solution of a difficult problem in analysis proceeds according to a general scheme: The given problem P with the solution S is replaced by a related problem P_n so simple that its solution S_n can be found with comparative ease. Then by improving the approximation P_n to P we may expect, or we may assume, or we may prove, that S_n tends to the desired solution S of P. The essential point in an individual case is to choose the sequence P_n in a suitable manner.

Here we shall deal mainly with problems of equilibrium and vibrations (boundary value and eigenvalue problems,[1] respectively). They lead to linear self-adjoint differential equations for an unknown function $u(x, y)$:

$$L(u) = f, \tag{1}$$

or

$$L(u) + \lambda u = 0, \tag{2}$$

in a two-dimensional domain of the x, y-plane, or rather to equivalent variational problems for the kinetic and potential energies of the system.

I. The variational problems

We assume our domain B bounded by a piecewise smooth curve C, and we denote the arc length measured along C by s, and differentiation in the direction of the inward normal by $\partial/\partial n$ or by a subscript n.

1. **Quadratic functionals.** Our variational problems refer to quadratic functionals

$$Q(v) = Q(v, v)$$

defined by symmetric bilinear expressions such as

$$D(v, w) = \iint_B (v_x w_x + v_y w_y)\,dx\,dy, \tag{3}$$

$$M(v, w) = \iint_B [\Delta v \Delta w + \alpha(v_{xx} w_{yy} + v_{yy} w_{xx} - 2v_{xy} w_{xy})]\,dx\,dy, \tag{4}$$

[1] Problems of critical loads (buckling) are likewise mathematically formulated as eigenvalue problems.

which occur in the study of membranes and plates, respectively. In (4) the symbol Δ denotes the Laplacian operator, and α a constant. Furthermore, we shall have to use other quadratic integrals defined by bilinear expressions such as

$$H(v, w) = \iint_B vw dx dy, \tag{5}$$

$$K(v, w) = \int_C vw ds, \tag{6}$$

$$R(v, w) = \int_L v_{xx} w_{xx} dx, \tag{7}$$

where L is a line $y = \text{const.}$ in B and C is the boundary of B. We then consider functionals such as

$$Q(v, w) = aD(v, w) + bM(v, w) + cK(v, w) + dR(v, w),$$

where a, b, c, d are constants. Always the *"admissible" functions* $u, v, w, \cdots$ are restricted by the condition that all the occurring integrands be at least piecewise continuous.

The stable equilibrium of a plate or membrane under an external pressure f is characterized by a variational problem of the type

$$Q(v) + 2H(v, f) = \text{minimum}, \tag{8}$$

for the deflection v, whereas vibrations of plates and membranes correspond to the problem of finding stationary values, $\nu^2 = \lambda$, of

$$Q(v)/H(v). \tag{9}$$

The values ν thus defined are the natural frequencies of the system. $Q(v)$ corresponds to the potential energy of the system in the case of equilibrium, while for vibrations $Q(v)$ and $H(v)$ are the "reduced" potential and kinetic energies, respectively.[2] Terms of the form K and R appearing in the expression Q represent *additional energies concentrated along the boundary* C of B or along a line L in B. For example, in the case of *reinforced plates* we would have $Q = M + dR$ if the reinforcement consists of a bar along the line L.

[2] By "reduced" energies we mean the following: If we assume the free system vibrating with a frequency ω (not necessarily a natural frequency), then the deflection u may be represented in the form $u = v \cos \omega t$, where v is a function of position only. The potential energy V may then be written in the form $Q(v) \cos^2 \omega t$ and the kinetic energy T in the form $\omega^2 H(v) \sin^2 \omega t$; the quantities Q and H are termed the reduced energies.

2. **Rigid and natural boundary conditions.** (See [1, 2, 9].) The Euler differential equations (1) or (2) of our variational problems must be supplemented by appropriate boundary conditions. If we focus our attention on the differential equations as such, it is not at all obvious what boundary conditions belong to a particular problem of mechanics. However, from the point of view of the calculus of variations a complete clarification of this delicate question is almost automatically obtained if the mathematical reasoning follows our mechanical intuition. In the first place we observe that in a variational problem (not so in a problem of differential equations) we need not in advance impose boundary conditions in order to single out a specific solution. If the functions admissible in the competition are not subjected to restrictions at the boundary, we speak of a "*free problem.*" In these problems the first variation of the functional will contain terms referring to the boundary, and the vanishing of the first variation will imply not only Euler's differential equation for the domain B but also conditions on the boundary C, which we call "*natural boundary conditions.*" Now the dominant fact is: *appropriate boundary conditions for differential equations are obtained as natural boundary conditions of corresponding variational problems.* In the latter they may, but they need not be prescribed in advance.

There is only the exceptional case, often termed the "simplest case of a variational problem" of *fixed boundary values* of u or derivatives of u or other expressions in u. Here the situation seems somewhat obscured. (The clamped membrane with the boundary condition $u=0$ and the clamped plate with the boundary conditions $u=u_x=u_y=0$ belong in this category.) Such fixed or *rigid* or *artificial boundary conditions* must be explicitly stipulated for the variational problem not only for the differential equation. However, we shall recognize them as limiting cases or degenerations of natural conditions.

Physically, rigid conditions correspond to rigid constraints of the system at the boundary C while natural conditions express equilibrium of the system of C if along C partial or full freedom of motion is permitted.

To understand the significance of natural boundary conditions the following observation is essential: The Euler differential equations depend only on the domain integrals or the energies spread over B. But the *natural* boundary conditions are essentially affected by the boundary integrals representing those contributions to the energies which are concentrated along the boundary C. These terms lead to a great variety of possible natural boundary conditions for the same differential equation. In a somewhat different way we may formulate

the fact: Natural boundary conditions, but not Euler's equations, are affected by divergence expressions in the domain integrals.

2a. **Examples. Plates, plane torsion for multiply-connected domains.** As an example we consider a membrane with a free boundary. A pressure f of average value zero may act on the membrane. In this case we have a variational problem for $D(v)+2H(v, f)$; if ζ denotes an arbitrary variation of v, the variational condition is

$$D(v, \zeta) + H(\zeta, f) = 0.$$

Transforming $D(v, \zeta)$ by Green's formula into a domain integral plus a boundary integral, we find the natural boundary condition

$$\frac{\partial v}{\partial n} = 0,$$

expressing the fact that no force acts on the boundary C. For a plate, free at the boundary, $Q(v) = M(v)$, and the natural boundary conditions appear as the classical Kirchhoff conditions:

$$(1 + \alpha)\Delta v = \alpha(v_{xx}x_n^2 + 2v_{xy}x_n y_n + v_{yy}y_n^2),$$

$$\frac{\partial}{\partial n}\Delta v = \alpha(v_{xx}x_n x_s + v_{xy}[x_n y_s + x_s y_n] + v_{yy}y_n y_s),$$

where x_n, y_n and x_s, y_s are, respectively, direction cosines of the inner normal and tangent vector along C.

In $M(v)$ the term $\alpha(v_{xx}v_{yy}-v_{xy}^2)=\alpha\partial(v_x v_{yy})/\partial x-\alpha\partial(v_x v_{xy})/\partial y$ is a divergence expression, very essential for Kirchhoff's natural boundary conditions but irrelevant for Euler's differential equation $\Delta\Delta v=0$ and without consequence for the clamped plate.

Similar remarks apply to *natural "discontinuity conditions"* that arise if energy is concentrated along lines L interior to B, such as in the case of *reinforced plates*. For example, for a rectangular plate clamped at the boundary but reinforced by a bar along the line $y=0$, the variational problem becomes

$$Q(v) + 2H(v, f) + kR(v) = \iint_B [(\Delta v)^2 + 2\alpha(v_{xx}v_{yy} - v_{xy}^2)]dxdy$$

$$+ k\int_L v_{xx}^2 dx + 2\iint_B vf dxdy = \min.$$

with the condition that $v=v_x=v_y=0$ on C and v, v_x, v_y are continuous in B while the second derivatives of v are at least piecewise continu-

ous in B. If ζ is an arbitrary variation of v satisfying these conditions, we obtain as above not only the Euler equation $\Delta\Delta v+f=0$, but also the *natural discontinuity condition*

$$[(\Delta v + \alpha v_{xx})]_{-}^{+} = \frac{\partial^4}{\partial x^4} v(x, 0),$$

where the symbol $[g]_{-}^{+}$ means the amount of discontinuity suffered by a function g in crossing the line $y=0$ from positive to negative values of y. Another example of a somewhat different character is the the problem of *torsion of long columns with multiply connected cross sections*. The contour C of the cross section may include a domain B in the x, y-plane from which are removed holes $B_1, B_2, B_3, \cdots$ with contours $C_1, C_2, C_3, \cdots$ and areas $A_1, A_2, A_3, \cdots$. The multiply-connected domain between C and $C_1, C_2, C_3, \cdots$ may be called B^*.

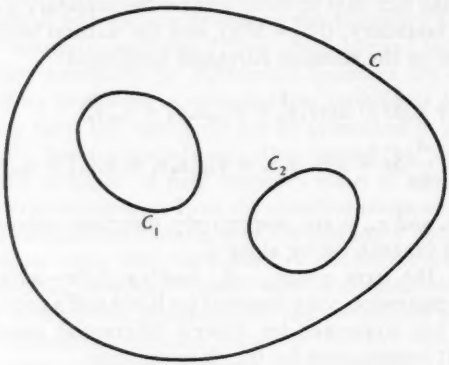

FIG. 1

Then the adequate variational formulation of the torsion problem in proper units is: To find a function $\phi=u$ continuous in $B+C$, having piecewise continuous first derivatives in B, having the boundary values zero on C and constant, but not prescribed values c_i in the holes B_i, such that for the whole domain B

$$D(\phi) = \iint [(\phi_x^2 + \phi_y^2) + 2\phi]dxdy$$

attains its least value d for $\phi=u$. The function u then will give the stresses in the cross section by differentiation.

Note that this problem requires not only to determine u in B^* as a solution of Euler's equation

$$\Delta u = 1$$

with the boundary conditions $u=0$ on C and $u=\text{const.}=c_i$ on C_i, but also to find the constants c_i. The freedom for these parameters in the functions ϕ admissible in the variational problem must lead to natural conditions from which the c_i are to be characterized.

Indeed, from the variational condition $D(u, \zeta)\times\iint\zeta dxdy=0$ valid for arbitrary ζ that vanishes on C and has constant values in each B: we can only obtain Euler's equation, but likewise immediately the natural boundary conditions

$$\int_{c_i} \frac{\partial u}{\partial n} ds + c_i A_i = 0$$

which connect the unknown boundary constants with the known areas A_i.

Incidentally, for the special choice $\zeta=u$ we obtain (in line with more general results) that the solution, representing a state of equilibrium, satisfies the relation

$$S = D(u) = -\iint u dxdy.$$

The quantity S represents the "*total stiffness*" of the column with respect to torsion.

In the appendix we shall see how this problem, which as an ordinary boundary value problem of a partial differential equation would be rather formidable, can be attacked numerically with success from the point of view developed here.

2b. **Rigid constraints as limiting cases.** If we have additional energy concentrated at the boundary and expressed, for example, by the term $K(v)=\int_C v^2 ds$, then not the Euler equation but the natural boundary conditions will be influenced by these terms. Thus, for the free membrane with

$$Q(v) = D(v) + \gamma K(v)$$

we obtain the natural boundary condition

$$\frac{\partial v}{\partial n} - \gamma v = 0. \tag{10}$$

We observe that as the parameter γ increases indefinitely, that is, as

the elastic restoring forces at the boundary approach a rigid constraint, the condition (10) tends to the boundary condition $v=0$ of the clamped membrane. This is in harmony with the fact that for large positive γ the boundary values of v^2 though free, must in the average be small in order to keep the energy $Q(v)$ in bounds.

Quite generally rigid boundary conditions should be regarded as limiting cases of natural conditions in which a parameter tends to infinity. This corresponds to the physical fact that *rigid constraints are only idealized limiting cases of very large restoring forces.*[3] It may be mentioned that this interpretation of fixed boundary conditions is the key to a more penetrating analysis of the question what we may and what we may not prescribe at the boundary (see [1]).

3. **General conclusions.** The mere formulation of our problem in terms of maxima and minima leads to further important applications. As an example we mention a famous *principle* first formulated by Lord Rayleigh [38]: If a vibrating system whose energies are expressed by quadratic integrals is changed into another system by an increase of masses or by a decrease of elastic forces, then all the natural frequencies of the system can only change toward lower values. This fact was recognized by the speaker (see [9]) as a consequence of the following theorem, which is easily proved: The nth natural frequency of any of our vibration problems is the *highest value of the lowest frequency* of all systems obtained from the given system by imposing $n-1$ constraints; or the nth eigenvalue λ_n is the largest value attained by the minimum $d(w_1, \cdots, w_{n-1})$ of the quotient $Q(v)/H(v)$ if v is subjected to $n-1$ linear conditions of the form

$$H(v, w_i) = 0, \qquad i = 1, \cdots, n-1,$$

the w_i being arbitrarily chosen functions. Here "minimum" refers to a fixed set of $n-1$ functions w_i.

This principle can render useful service for appraising the change in the natural frequencies resulting from changes in the given vibrating system. We recognize immediately that a stiffening of the system by the introduction of new elastic forces (which lead to an increase in potential energy) must produce higher natural frequencies throughout. Likewise, imposing new rigid constraints will have the same effect. Even quantitative results can be obtained by a simple applica-

[3] It might be mentioned that A. Weinstein's method (see §3) is a somewhat different way of presenting rigid boundaries as limiting cases of problems with fewer restrictions.

tion of this principle. The asymptotic behavior of the natural frequencies $\nu_n = \lambda_n^{1/2}$ for large n was found by A. Sommerfeld, H. Weyl, and the speaker. For example, the nth eigenvalue of $\Delta v + \lambda v = 0$ for a free or clamped boundary and domain of area A behaves asymptotically like $4\pi n/A$. (See [9].)

Recently Alexander Weinstein has made an interesting application of this maximum-minimum principle to clamped plates [4]. The eigenvalue problem for a "supported" plate (with $\alpha = 0$) refers to the quadratic energy expressions $Q(v) = \iint_B (\Delta v)^2 dxdy$, $H(v) = \iint_B v^2 dxdy$. It is then required to make the quotient Q/H stationary only under the boundary condition $v = 0$. From this variational problem, we obtain the Euler equation $\Delta\Delta v - \lambda v = 0$ and one natural boundary condition, $\Delta v = 0$; the solutions for this problem are in this case identical with those of the clamped membrane. Now Weinstein, by imposing successively a denumerable number of boundary conditions of the form $\int_C (\partial v/\partial n)\phi_i ds = 0$, $i = 1, 2, 3, \cdots$, where the ϕ_i form a complete system of functions on the boundary C, obtains the problem of the clamped plate as a limiting case. The approximating problems stipulate conditions less restrictive than the limiting problem of a clamped plate, consequently they lead to smaller values of the minima and hence of the maxi-minima, and therefore provide lower bounds for the natural frequencies of the clamped plate. It is remarkable that the approximating problems could be solved explicitly in terms of solutions of the membrane problem [20, 21, 22].

II. Rayleigh-Ritz method

We now discuss the question of attacking a variational problem numerically. In principle, there are many ways in which such a variational problem may suggest approximations by simpler problems. The Rayleigh-Ritz method is only one of them.

1. **The principle and theoretical aspects.** Suppose we seek the minimum d of an integral expression or any other variational expression $I(\phi)$ (for example, our quadratic functionals of the preceding section). We then start with a *minimizing sequence*

$$\phi_1, \phi_2, \phi_3, \cdots, \phi_n, \cdots, \tag{11}$$

that is, a sequence of functions, admissible in our variational problem, for which

$$\lim_{n\to\infty} I(\phi_n) = d, \tag{12}$$

d being the lower bound of the functional $I(\phi)$. The existence of the lower bound d is obvious or may be easily proved in all relevant problems and the *existence* of the minimizing sequence (11) is then a logical consequence.

However, the problem in applications is one, not of the existence, but of the practical *construction* of such a minimizing sequence. Ritz's method is nothing but a recipe for such a construction. A minimizing sequence immediately furnishes an approximation to d (sometimes this is all we wish to know, for example, if we are interested in the natural frequencies of a vibrating system). Moreover, it may be assumed and in many cases it can be proved that the minimizing sequence itself will furnish a good approximation for the function u which actually solves the problem.

Ritz's construction proceeds as follows: We start with an arbitrarily chosen system of *"coordinate functions"*

$$\omega_1, \omega_2, \cdots, \omega_n, \cdots, \tag{13}$$

which should satisfy the two conditions:

(a) Any linear combination

$$\phi_n = c_1\omega_1 + c_2\omega_2 + \cdots + c_n\omega_n \tag{14}$$

of them is admissible in the variational problem.

(b) They should form a complete system of functions in the sense that *any* admissible function ϕ and its relevant derivatives may be approximated with any degree of accuracy by a linear combination of coordinate functions and of their corresponding derivatives, respectively.

If we begin with such a system of coordinate functions, it is clear that for n sufficiently large and for a suitable choice of the coefficients $c_1, c_2, \cdots, c_n$ of (14) we can find admissible functions ϕ_n for which $I(\phi_n)$ differs arbitrarily little from d. In other words, it is possible to find a minimizing sequence $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ as a sequence of linear combinations of the coordinate functions. In order to obtain such a minimizing sequence we choose the c_i in the following manner. We consider any function ϕ_n defined by (14) and substitute it in our variational problem. $I(\phi_n)$ then becomes a function $F(c_1, \cdots, c_n)$ of the n parameters c_i which we may now determine from the ordinary minimum problem of the calculus

$$I(\phi_n) = F(c_1, \cdots, c_n) = \min. \tag{15}$$

In the problems considered here, $I(\phi)$ is a quadratic or bilinear functional, and (15) leads to a system of n linear equations for the

parameters,[4] a system which may be solved by established methods. Thus the minimizing sequence, ϕ_n $(n=1, 2, \cdots)$ is found. Recalling our general scheme we may identify the construction of the ϕ_n with the problem P_n.

At this point one important remark may be made [1]. While the convergence of $I(\phi_n)$ to d is assured, it is by no means generally true that ϕ_n tends to u, the solution of the original minimum problem, or, even less so, that the derivatives of ϕ_n tend to the corresponding derivatives of u. However, a comparative investigation of different types of variational problems reveals that, generally speaking, the *convergence* of a minimizing sequence $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ and the sequences of the derivatives of the ϕ_n is *improved if the order of the occurring derivatives becomes higher*. On the other hand, there is a tendency toward *worse convergence as the number of independent variables increases*. For example, in the one-dimensional problem of an elastic string $(Q(v)=\int_0^l v'^2 dx)$, the convergence of the ϕ_n to u is assured, the derivatives ϕ_n', however, need not converge to u'. But for the corresponding problem of the bar $(Q(v)=\int_0^l v''^2 dx)$, not only does ϕ_n converge to u but also ϕ_n' to u'. On the other hand, in the membrane problem even the ϕ_n need not converge to u, while in the case of the plate the convergence of ϕ_n to u is assured. The first success attained by Ritz depended largely on his good fortune in attacking the seemingly more difficult problem of the plate rather than that of the membrane.

These facts which are intimately related to more profound questions in the general theory of the variational calculus have suggested the following method of obtaining better convergence in the Rayleigh-Ritz method. Instead of considering the simple variational problem for the corresponding boundary value problem, we modify the former problem without changing the solution of the latter. This is accomplished by adding to the original variational expression terms of higher order which vanish for the actual solution u. For

[4] This suggests the following interpretation and generalization of the procedure whereby reference to a minimum problem need no longer be made: we replace our differential equation $L(u)=0$ by the condition that $L(u)$ should be orthogonal to n functions of a previously selected complete system of functions. If, then, n tends to infinity, the totality of all these relations will be substituted for the differential equation, and for any fixed n, u may be chosen, for example, as a linear combination (14): $u=\phi_n$. In this general pattern that goes back to Galerkin, there is more freedom left for the choice of the approximations to the solution u. However, in the Rayleigh-Ritz method proper, generally speaking the question of convergence is more easily investigated. (The generalized *view* interpretation is implicitly mentioned in Ritz' papers and was later developed by several authors. See for example [18, 19].)

example, we may formulate the equilibrium problem for a membrane under the external pressure f as follows:

$$I(v) = \iint_B (v_x^2 + v_y^2 + vf)dxdy + \iint_B k(\Delta v - f)^2 dxdy = \text{min.},$$

where k is an arbitrary positive constant or function. Such additional terms make $I(v)$ more *sensitive* to variations of v without changing the solution. In other words, minimizing sequences attached to such a "sensitized" functional will by force behave better as regards convergence [7].

The practical value of the method of sensitizing the integral by the addition of terms of higher order has not yet been sufficiently explored. Certainly the sensitizing terms will lead to a more complicated system of equations for the c_i. This means that a compromise must be made for a suitable choice of the arbitrary positive function k so that good convergence is assured while the necessary labor is kept within bounds.

2. **Practical viewpoints.** Theoretically the Rayleigh-Ritz method consists merely in the construction of the minimizing sequence. However, the difficulty that challenges the inventive skill of the applied mathematician is to find suitable coordinate functions and to estimate the accuracy of the result. From a practical point of view almost any success depends on the selection of coordinate functions. If these functions are chosen without proper regard for the individuality of the problem the task of computation will become hopeless. A choice should be made so that the system of linear equations for the c_i obtained from (15) will have a preponderance of terms along the diagonal of their matrix, and that the number of terms to be taken into account be kept small. Since only a few of the coordinate functions will enter into the calculation, the theoretical completeness of the ϕ_n is irrelevant. It is important that the initial function be already a fair approximation, and furthermore that the functions ϕ_1, ϕ_2, $\cdots$ should be sufficiently different[5] so that increasing the number of terms will lead to an actual improvement of the approximation. Importance should also be attached to the need for making the numerical evaluation of the coefficients of our linear equations practicable.

In many cases the use of polynomials for coordinate functions is most advantageous. Si Luan Wei has shown in his thesis that results

[5] Such a "difference" can be measured. See Courant-Hilbert, *Methoden der mathemateschen Physik*, vol. I, p. 52.

for plates can be obtained by polynomials more readily and more accurately than by the coordinate functions originally used by Ritz [40].

Recently physicists have chosen coordinate functions according to the following pattern. One starts with a choice of the function ω_1, which is expected to be a fair approximation to the actual solution. Then ω_2 is defined by $\omega_2 = L(\omega_1)$. Finally one chooses $\omega_3 = L(\omega_2) = LL(\omega_1)$, $\omega_4 = L(\omega_3) = LLL(\omega_1)$, and so on. This choice makes the calculation of the matrix elements of our linear equations for the c_i comparatively simple. Since the operator L introduces higher derivatives which may lead to cumbersome complications at boundaries, it is understandable that such a choice of the ω_n seems feasible primarily for infinite domains where no boundary is given and where the natural boundary conditions are equivalent to the finiteness of the variational integrals.

3. **Boundary conditions.** For rigid boundary conditions the approximation by the Rayleigh-Ritz method is comparatively good. Few admissible coordinate functions would in most cases suffice to yield a result near the desired solution. This pleasant feature, however, is offset by the restriction imposed by the rigid boundary conditions which in general precludes the choice of simple coordinate functions.

For free boundaries and natural boundary conditions the choice of coordinate functions is eased considerably since no boundary conditions need be stipulated in advance. As a general rule we might well use, for the ϕ_n, polynomials with undetermined coefficients. For this great advantage, however, we must pay a price, namely, the necessity for using many more terms to secure reasonable accuracy. Therefore it is sometimes preferable if we can satisfy in advance, at least approximately, the natural boundary conditions, by a proper choice of functions. Still, the advantage gained by making the ϕ_n polynomials might be decisive.

In this connection it seems to be of importance that *rigid boundaries can be considered as a limiting case of free boundaries*, as indicated previously in §I, 2. As an example we may consider the following problem for a membrane:

$$D(v) + 2H(v, f) + \gamma K(v) = \iint_B (v_x^2 + v_y^2 + 2vf)dxdy + \gamma \int_C v^2 ds = \min.$$

If γ is very large, then the free boundary problem relating to this expression is almost identical to the corresponding problem for a clamped membrane. By letting γ tend to infinity we "*freeze*" the boundary. This suggests an attempt to solve the equilibrium problem of the clamped plate by choosing a large numerical value for γ and then treating the problem as one for a free membrane. However, to obtain reasonable accuracy for the free problem, the larger the quantity γ the more terms will be necessary. Hence the practical application of the method again requires a compromise—γ must be chosen large enough to approximate rigidity but small enough to keep the necessary labor within reasonable bounds. From a theoretical as well as from a practical point of view it would seem worth while to study the preferable choices of these artificial parameters.

4. **Estimates.** A weak point in the Rayleigh-Ritz procedure is that it does not contain a principle for estimating the accuracy of the approximation. This is not the place to give an account of the numerous efforts made to fill this gap at least theoretically. For problems of equilibrium, estimates for the minimum d can be obtained by a method suggested by Castigliano's principle in the theory of elasticity. The mathematical idea is to represent the minimum value d of the given problem as a maximum value of another variational problem (see [28]). The margins obtained are quite narrow. But the practical value of this accuracy does not appear great since the value of d in problems of equilibrium is of little interest,[6] whereas we are concerned with the deflection u and its derivatives. In the case of vibrations, where the minimum values, as squares of the frequencies, are of considerable importance, methods for the estimation of accuracy are in general less satisfactory [29].

5. **Objections to the Rayleigh-Ritz method.** The vagueness as to the accuracy of the approximation obtained is only one of the objections to the Rayleigh-Ritz method that may be raised. More annoying is that a suitable selection of the coordinate functions is often very difficult and that laborious computations are sometimes necessary. For these reasons, alternative methods must be studied.

III. Method of finite differences. General random statistical methods

As far as practical experience goes the most important of these

[6] See, however, E. Trefftz, Math. Ann. vol. 108 (1933), p. 595 where the deflection at a point x_0, y_0 is represented as such a minimum d of a modified functional.

methods is that of finite differences. In this well known procedure we replace differential quotients by difference quotients and integral expressions by finite sums defined over a set of net-points in the plane, for example, a quadratic net in the domain B formed by the intersection of the lines $x=\gamma h$, $y=\mu h$ $(\gamma=1,2,\cdots,N;\mu=1,2,\cdots,M)$. The simplified problem P_n of difference equations can very often be solved with relative ease. Then if we permit the mesh, $h=1/n$, of the net to tend to zero with increasing n, not only does P_n tend to P, but the solutions S_n of the difference equations approach the solution S of the original problem exceedingly well. Furthermore, we have the remarkable fact that all relevant difference quotients of first and higher order converge to the corresponding derivatives pertaining to the original problem.

On these grounds the method of finite difference as a general procedure is often preferable to the Rayleigh-Ritz method. The latter might lend itself more readily to the solution of specific problems where suitable analytic expressions are available for coordinate functions. However, in other cases experience points to the superiority of the method of finite differences. One of the underlying reasons is that finite differences are attached directly to the values of the function itself without an interceding medium such as the more or less arbitrary coordinate functions.

If the variational problems contain derivatives not higher than the first order the method of finite difference can be subordinated to the Rayleigh-Ritz method by considering in the competition only functions ϕ which are linear in the meshes of a sub-division of our net into triangles formed by diagonals of the squares of the net. For such polyhedral functions the integrals become sums expressed by the finite number of values of ϕ in the net-points and the minimum conditions become our difference equations. Such an interpretation suggests a *wide generalization* which provides great flexibility and seems to have considerable practical value. Instead of starting with a quadratic or rectangular net we may consider from the outset any polyhedral surfaces with edges over an *arbitrarily chosen* (preferably triangular) *net*. Our integrals again become finite sums, and the minimum condition will be equations for the values of ϕ in the net-points. While these equations seem less simple than the original difference equations, we gain the enormous advantage of better adaptability to the data of the problem and thus we can often obtain good results with very little numerical calculation. (See appendix.)

This procedure of finite differences was analyzed from a mathematical point of view—and in particular the convergence for the quad-

ratic mesh width $h \to 0$ was proved—first by Philipps in unpublished papers, then by Philipps and Wiener [14], and at the same time by Lusternik and by Courant [5], and later by Courant, Friedrichs, and Lewy [8].

As was already observed by Philipps and Wiener, equations of finite differences can be interpreted by processes of probability—the so-called random walk. This connection has led Courant, R. Lueneburg [16], and Petrovsky [17] to a more general attack upon our problems, as follows: Instead of simply replacing our integral expressions by finite sums defined over a net, we may replace them in various ways by other simple functionals and thus obtain a greater variety of possible approximate problems P_n. Consider, for example, the non-negative function $K(x, y; \xi, \eta)$ which is assumed to be symmetric, to be defined and piecewise continuous in the entire plane and to satisfy $\int\int_{-\infty}^{+\infty} K(x, y; \xi, \eta)\, d\xi d\eta = 1$. We then focus our attention on integrals of the following form

$$(16)\quad T(\omega) = \int\int\int\int_{-\infty}^{+\infty} K(x, y; \xi, \eta)[\omega(x, y) - \omega(\xi, \eta)]^2 dx dy d\xi d\eta$$

taken over the entire plane, and investigate the problem P_n,

$$(16a)\qquad T(\omega) = \min.,$$

for which the admissible functions are prescribed in the domain $\overline{B}$, complementary to B, that is, in the portion of the plane outside of B.

For example, if $K = f(r)$, where $r^2 = (x-\xi)^2 + (y-\eta)^2$ and $f(r) \geqq 0$, while $f(r) = 0$ for $r > h$, it is easily verified that, as the parameter h tends to zero, we have $T(\omega) \to D(\omega)$ so that P_n tends to the boundary value problem for $\Delta u = 0$. It can be shown generally that the nucleus K may be chosen dependent on a parameter h so that $T(\omega)$ tends to a given quadratic functional which yields as Euler equation any prescribed homogeneous elliptic differential equation.

Now the variational condition for the problem (16a) is no longer a differential equation but a Fredholm integral equation, which is more easily treated:

$$(17)\qquad u(x, y) = \int\int_B K(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta + g(x, y),$$

where

$$g(x, y) = \int\int_{\overline{B}} K(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta$$

is known because u is prescribed in $\overline{B}$. The solution u_n of (17) for $K=K_n$ converges to the solution u of the problem P. Moreover, and this is the important point for practical applications, it can be proved that the integral equation (17) is solvable by the Neumann method of iteration under very wide conditions for the nucleus K. Thus we may develop a new method for attacking boundary value problems which, however, so far has not been tested in practical applications.

It may be remarked that, generally speaking, the iteration process will converge more slowly if the approximation of P_n to P becomes better. Hence again a compromise for practical purposes is indicated.

In passing, we mention the intimate *connection* of this method *with statistics*. Let us suppose a substance to be distributed with a density $u(x, y)$ over the plane. We now imagine that the substance is redistributed in distinct steps in such a manner that a unit mass concentrated at the point $P(x, y)$ will be spread over the plane with a density $K(x, y; \xi, \eta)$ at the point $Q(\xi, \eta)$. The integral equation (17) then characterizes a state of statistical equilibrium, if the density $u(x, y)$ is prescribed in the complementary domain $\overline{B}$ of B. Of course, if we interpret the integrals as Stieltjes' integrals, we may include in our formulation even problems of finite differences and random walk problems of the classical type. For us here, the main objective is to point out a method that enables us in principle to find approximately solutions of boundary value problems.

IV. Method of gradients

Still another alternative to the Rayleigh-Ritz procedure should be mentioned. This is the method of gradients, which goes back to a paper published by Hadamard in 1907 [12]. Highly suggestive as is Hadamard's attempt, difficulties of convergence were encountered. However, recent developments in the theory of conformal mapping and in Plateau's problem throw new light on Hadamard's idea, so that it seems justified now to expect from it not only purely mathematical existence proofs but also a basis for numerical treatment in suitable cases [13, 10, 11].

The principle of the method may be understood from the elementary geometric concept of a vector gradient. Let $u=f(x_1, \cdots, x_n)$ be a non-negative function of the n variables x_i, or as we might say of the position vector $X=(x_1, \cdots, x_n)$, and let us seek to determine a vector $X=X_0$ for which u is at least stationary. We then proceed as follows: on the surface $u=f(x)$ we move a point $(x_1, \cdots, x_n, u)$ so that $x_i(t)$ and $u(t)$ become functions of a time-parameter t. Then

the velocity of ascent or descent along the line $X=X(t)$, $u=u(t)$ on the surface is

$$\frac{du}{dt} = \dot{u} = \sum_{i=1}^{n} \dot{x}_i f_{x_i} = \dot{X} \cdot \operatorname{grad} f.$$

We now choose the line along the line along which the motion proceeds so that the descent is as steep as possible (*lines of steepest descent*). This means to make u negative and as large "as possible" in absolute value, for example, by choosing

$$\dot{X} = -\operatorname{grad} f, \tag{18}$$

so that

$$\dot{u} = -(\operatorname{grad} f)^2.$$

Hence the position vector X moves according to the system of ordinary differential equations (18) along the lines of steepest descent with respect to the function f. Under very general assumptions, it is clear that X, starting from an *arbitrary* initial position, will, for $t \to \infty$, approach a position for which $\operatorname{grad} f = 0$, and therefore for which f is stationary and possibly a minimum. However, instead of using the continuous procedure given by the differential equation (18), we may proceed stepwise, correcting a set of approximations x to the solutions of the equations $\operatorname{grad} f = 0$ by corrections proportional to the respective components of $-\operatorname{grad} f$.

This elementary idea can be generalized to variational problems. If we wish to determine a function $u(x, y)$ defined in B and having prescribed boundary values such that u is the solution of a variational problem

$$I(v) = \iint_B F(x, y, v, v_x, v_y)\,dx\,dy = \min., \tag{19}$$

then we interpret the desired function u as the limit for $t \to \infty$ of a function $v(x, y, t)$, whose values may be chosen arbitrarily for $t=0$ and for all t thereafter are determined in such a way that the expression $I(v)$, considered as a function $I(t)$ of t, decreases as rapidly as possible toward its minimal value. Of course the boundary values of $v(x, y, t)$ are the same as those for $u(x, y)$, so that v_t must vanish at the boundary. If we choose $v = v(x, y, t)$, we find

$$\dot{I}(t) = -\iint_B v_t L(v)\,dx\,dy, \tag{20}$$

where $L(v)$ is the Euler expression corresponding to (20). To consider a concrete example, we suppose that

$$I(v) = \iint_B (v_x^2 + v_y^2)dxdy, \tag{21}$$

so that our minimum problem amounts to determining the equilibrium of a membrane with given boundary deflections $g(s)$. Then $L(v) = -2\Delta v$. Incidentally (20) displays an analogy between the Euler expression and the gradient of a function $f(x_1, \cdots, x_n)$ of n independent variables. The variation or "velocity" of $I(v)$ is expressed as an "inner product" of the velocity of the "independent function" v with the Euler expression $L(v)$, the *gradient of a functional in function space.*

We now assure ourselves of a steady descent or decrease of $I(t)$ by choosing v_t in accordance with the differential equation

$$v_t = -kL(v), \tag{22}$$

where k is a positive arbitrary function of x, y. (21) then becomes

$$I(t) = -\iint_B k[L(v)]^2 dxdy,$$

and again we can infer that, for $t \to \infty$, $v(x, y, t)$ will tend to the solution $u(x, y)$ of the corresponding boundary value problem $L(u) = 0$.

For the case of the membrane the differential equation (22) becomes

$$v_t = \Delta v, \tag{23}$$

the equation of *heat transfer*. In our interpretation this equation describes a rapid approach to a stationary state along the "lines of steepest descent." While for the equations (23) or (22) the convergence of v for $t \to \infty$ can be proved, serious difficulty arises if we want to replace our continuous process by a process of stepwise corrections as would be required for numerical applications. Each step means a correction proportional to Δv, thus introducing higher and higher derivatives of the initial function v. Another great difficulty is presented by rigid boundary values.[7]

Yet there do exist classes of problems where such difficulties can be overcome if the method is extended properly. First of all we may

[7] Incidentally, if we apply this procedure to a problem for a finite net, it converges very well and is, as a matter of fact, nothing but a natural method of solving a system of linear equations by a method of iteration.

observe that it is not necessary to select the *steepest* descent along the gradient; it suffices to secure a *safe* descent at a suitably fast rate. Furthermore, if we consider problems for which the boundary value problem of the differential equation presents no difficulties for the domain B, but for which a degree of freedom in the boundary values is left, then the problem reduces to one for finding these boundary values, and now all our difficulties disappear. A typical example is the problem of the conformal mapping of a circle B onto a simply-connected domain G, and it may be that the method of gradients opens a path for the attack of the problem of conformal mapping for multiply-connected domains and other problems as well.

This address has emphasized more theoretical aspects. However, some of the general principles described may be helpful in enlarging our equipment for practical purposes.

Appendix[8]

Numerical treatment of the plane torsion problem for multiply-connected domains

The computation of the stiffness S defined in §I, 2a furnishes an example of independent interest which permits to compare the practical merits of some of the methods described in this address. Numerical calculations were carried out for the cross sections of the following diagrams, a square from which a smaller square is cut out; and a square, from which four squares are cut out. In the first case our quadratic frame was supposed to be bounded by the lines $x = \pm 1$, $y = \pm 1$ and $x = \pm 3/4$, $y = \pm 3/4$. To apply the Rayleigh-Ritz method for the domain as a whole would already be cumbersome because of the boundary conditions for admissible functions ϕ. However, this difficulty disappears if we exploit the symmetry of the domain and the resulting symmetry of the solution; thus we may confine ourselves to considering only one-eighth of the domain B^*, namely the quadrangle $ABCD$. For this polygon any function of the type

$$\phi = a(1 - x)[1 + (x - 3/4)P]$$

where $P(x, y)$ is a polynomial, is admissible, and its substitution in the integral leads to simple linear equations for the cofficients. Thus for the simplest attempt

$$\phi = a(1 - x)$$

which leaves only one constant a to be determined, we find with a

[8] Addition not contained in the original address.

negligible amount of numerical labor $S=.339$ and $c=-.11$. A refined attempt with the function

$$\phi = a(1-x)[1+\alpha(x-3/4)y]$$

yielded $S=.340$ and $c=-.109$ with little more labor.

These results were checked with those obtained by our generalized method of finite differences where arbitrary triangular nets are permitted. The diagrams are self-explanatory. Unknown are the

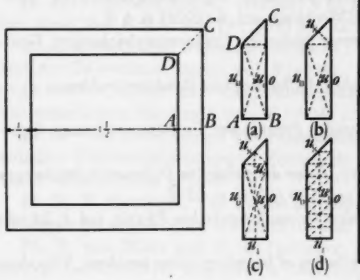

FIG. 2

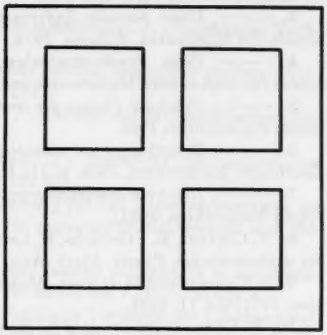

FIG. 3

net-point-values u_i, $(c=u_0)$. In the net-triangles our functions were chosen as linear, so that the variational problem results in linear equations for the u_i. The results, easily obtainable, were: case (a) with two unknowns: $S=.344$, $u_0=-.11$; case (b) with three unknowns: $S=.352$, $u_0=-.11$; case (c) with five unknowns $S=.353$, $u_0=-.11$; case (d) with nine unknowns, corresponding to the ordinary difference method $S=.353$, $u_0=-.11$.

These results show in themselves and by comparison that the generalized method of triangular nets seems to have advantages. It was applied with similar success to the case of a square with four holes, and it is obviously adaptable to any type of domain, much more so than the Rayleigh-Ritz procedure in which the construction of admissible functions would usually offer decisive obstacles.

In a separate publication it will be shown how the method can be extended also to problems of plates and to other problems involving higher derivatives.

Of course, one must not expect good local results from a method

using so few elements. However, it might be expected that a smooth interpolation of the net functions obtained will yield functions which themselves with their derivatives are fairly good approximations to the actual quantities.

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NEW YORK UNIVERSITY

THE OCTOBER MEETING IN NEW YORK

The three hundred ninetieth meeting of the American Mathematical Society was held at the Hotel Pennsylvania on Friday and at Columbia University on Saturday, October 30–31, 1942. The attendance included the following one hundred fifty-eight members of the Society:

A. A. Albert, T. W. Anderson, R. G. Archibald, L. A. Aroian, S. F. Barber, J. L. Barnes, A. E. Basch, M. F. Becker, Stefan Bergman, P. G. Bergmann, E. E. Betz, Felix Bernstein, Garrett Birkhoff, G. D. Birkhoff, Gertrude Blanch, Salomon Bochner, H. F. Bohnenblust, S. G. Bourne, Joseph Bowden, H. W. Brinkmann, G. W. Brown, Hobart Bushey, J. H. Bushey, S. S. Cairns, M. E. Carlen, M. D. Clement, I. S. Cohen, Harvey Cohn, T. F. Cope, Richard Courant, J. E. Crawford, M. T. Curran, J. H. Curtiss, Jesse Douglas, Arnold Dresden, Nelson Dunford, J. E. Eaton, M. L. Elveback, Paul Erdös, A. D. Fialkow, M. M. Flood, Tomlinson Fort, M. C. Foster, R. M. Foster, K. O. Friedrichs, B. P. Gill, L. M. Graves, Bernard Greenspan, C. C. Grove, E. J. Gumbel, Margaret Gurney, Jacques Hadamard, F. C. Hall, L. A. Hazeltine, E. R. Hedrick, Edward Helly, Aaron Herschfeld, Max Herzberger, E. H. Hildebrandt, Einar Hille, Abraham Hillman, Banesh Hoffmann, T. R. Hollcroft, M. W. Hopkins, E. M. Hull, R. P. Isaacs, S. A. Joffe, Fritz John, Edward Kasner, J. R. Kline, B. O. Koopman, Mark Kormes, Arthur Korn, H. N. Laden, M. E. Ladue, J. A. Larrivee, Solomon Lefschetz, B. A. Lengyel, Marie Litzinger, E. R. Lorch, A. N. Lowan, R. K. Luneberg, A. W. McMillan, Brockway McMillan, L. A. MacColl, C. C. MacDuffee, Saunders MacLane, H. F. MacNeish, H. B. Mann, A. J. Maria, M. H. Maria, Imanuel Marx, Walther Mayer, A. E. Meder, H. L. Mintzer, E. C. Molina, Parry Moon, Marston Morse, G. W. Mullins, C. A. Nelson, C. O. Oakley, L. F. Ollmann, Oystein Ore, J. C. Oxtoby, E. L. Post, R. G. Putnam, Hans Rademacher, J. F. Randolph, L. L. Rauch, Mina Rees, Moses Richardson, J. F. Ritt, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Arthur Sard, R. D. Schafer, Henry Scheffé, M. G. Scherberg, I. E. Segal, Abraham Seidenberg, Stephan Serghiesco, Bernard Sherman, Seymour Sherman, Max Shiffman, James Singer, R. R. Singleton, P. A. Smith, Ernst Snapper, A. H. Sprague, E. R. Stabler, J. J. Stoker, W. C. Strodt, S. R. R. Struik, M. M. Sullivan, J. L. Synge, J. D. Tamarkin, W. R. Transue, C. A. Truesdell, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, D. F. Votaw, L. I. Wade, W. R. Wasow, Louis Weisner, M. E. Wells, H. S. White, A. L. Whiteman, P. M. Whitman, J. E. Wilkins, S. S. Wilks, John Williamson, H. A. Wood, Alexander Wundheiler, Leo Zippin, O. J. Zobel, Antoni Zygmund.

Friday afternoon the Society joined with the Optical Society of America in a symposium on *Mathematics in the field of optics*. Mr. R. M. Foster presided at the session which included the following addresses: *The theory of the Schmidt telescope* by Professor J. L. Synge of the University of Toronto, *The mathematics of turbid media* by Dr. S. Q. Duntley of the Massachusetts Institute of Technology, *A matrix treatment of optical systems, with polarizing, birefringent, and optically active elements* by Dr. R. C. Jones of the Bell Telephone

Laboratories, and *New methods of calculating illumination* by Professor Parry Moon of the Massachusetts Institute of Technology.

The Society met in two sections on Saturday morning, one for presentation of papers in analysis and geometry with Professor Antoni Zygmund presiding, and another for algebra, statistics and applied mathematics with Professor H. W. Brinkmann presiding.

President Marston Morse presided at the general session Saturday afternoon at which Professor Salomon Bochner of Princeton University gave an address on *Continuation of analytic functions in several variables.*

Members of the Society joined with members of the Optical Society of America and the Society of Rheology at lunch at the Hotel Pennsylvania on Friday. Dr. F. B. Jewett, Vice President of the American Telephone and Telegraph Company, was the speaker at the luncheon. Members of the Society were invited to attend the annual dinner of the Optical Society of America which was held at the Hotel Pennsylvania Friday evening. One of the speakers was President Marston Morse who discussed the interest of mathematicians in applied mathematics.

The Council met on Saturday, October 31, at 12:00 M. in the Men's Faculty Club at Columbia University.

The Secretary announced the election of the following three persons to membership in the Society:

Mr. Robert Steiner Fouch, Army Air Forces Pre-Flight School, Maxwell Field, Ala.;
Mr. Paul W. Healy, Case School of Applied Science;
Mr. Alvin Spiro, Ann Arbor, Mich.

The following appointments by President Marston Morse were reported: as a Committee on Printing Contracts, Dean Tomlinson Fort (chairman), Dean M. H. Ingraham, Professor J. D. Tamarkin; as representative of the Society at the celebration of the Fiftieth Anniversary of the Founding of The Woman's College of the University of North Carolina on October 4–5, 1942, Professor J. M. Thomas; as representative of the Society at the inauguration of John Nelson Russell Score as President of Southwestern University (Georgetown, Texas) on October 6, 1942, Professor E. L. Dodd; as representative of the Society at the inauguration of Charles Albert Anderson as President of Coe College (Cedar Rapids, Iowa) on November 12, 1942, Professor E. W. Chittenden.

The Secretary reported that Professor Harry Bateman had accepted the invitation to deliver the Gibbs Lecture at the Annual Meeting of 1943; the title of the Lecture will be *The control of elastic*

fluids. Professor Einar Hille has accepted the invitation to deliver the Colloquium Lectures at the Summer Meeting of 1944 on the following subject: *The analytical theory of semi-groups*.

It was announced that Professor Nathan Jacobson would give an invited address at the February, 1943, meeting in New York City, entitled *Some topics in the theory of semi-linear transformations*.

The Secretary reported that Professor F. D. Murnaghan had been named chairman of the temporary editorial committee for Mathematical Surveys and that Professor T. H. Hildebrandt had been appointed a member of the committee, to replace Professor E. J. McShane.

Professor E. J. Moulton and Dean W. G. Simon were appointed representatives on the Council of the American Association for the Advancement of Science for 1943.

Professors T. H. Hildebrandt, Hassler Whitney, and G. T. Whyburn were appointed representatives on the Editorial Board of the Annals of Mathematics for a period of three years beginning with 1943.

The Council appointed Professor A. D. Michal of California Institute of Technology as Associate Secretary in the far west, to fill the unexpired term of Dean T. M. Putnam.

At the Summer Meeting of 1942 a committee consisting of Professors O. E. Neugebauer and G. B. Price was appointed to investigate the possibilities of scientific communication with other countries. A report of this committee on the Russian situation was published in Volume 48, Number 11, of this Bulletin. The committee is continuing its investigations, with particular emphasis on communication with the Indian and Australian mathematicians.

Titles and cross references to the abstracts of papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1–2 were read in the section for analysis and geometry, those numbered 3–8 in the section for algebra, statistics and applied mathematics; those numbered 9–27 were read by title. Paper 2 was read by Professor Kasner, and paper 7 by Dr. Bers. Dr. Kogbetliantz was introduced by Professor J. A. Shohat, and Dr. Wong by Professor T. R. Hollcroft.

1. J. E. Wilkins: *The first canonical pencil.* (Abstract 49-1-82.)

2. Edward Kasner and John DeCicco: *A generalized theory of dynamical trajectories.* (Abstract 48-11-329.)

3. Ernst Snapper: *The resultant of a linear set.* (Abstract 48-11-309.)

4. R. P. Isaacs: *Application of polygenic functions to two-dimensional elasticity.* (Abstract 48-11-328.)

5. Garrett Birkhoff: *A reversibility paradox in hydrodynamics.* (Abstract 48-11-325.)

6. H. B. Mann: *The construction of orthogonal Latin squares.* (Abstract 48-11-336.)

7. Lipman Bers and Abe Gelbart: *On solutions of the differential equations of gas dynamics.* I. (Abstract 48-11-324.)

8. Max Herzberger: *Direct methods in geometrical optics.* (Abstract 49-1-64.)

9. R. H. Cameron and W. T. Martin: *Infinite linear difference equations with arbitrary real spans and first degree coefficients.* (Abstract 48-9-257-*t*.)

10. Nathaniel Coburn: *The linear yield condition in the plane plasticity problem.* (Abstract 48-11-326-*t*.)

11. John DeCicco: *New proofs of the theorems of Beltrami and Kasner on linear families* (Abstract 48-9-274-*t*.)

12. C. J. Everett: *Sequence completion of lattice moduls.* (Abstract 48-11-302-*t*.)

13. Abraham Hillman and H. E. Salzer: *Complex roots of* $\sin z = z$. (Abstract 48-11-314-*t*.)

14. Irving Kaplansky: *Systems of congruences in a valuation ring.* (Abstract 48-11-303-*t*.)

15. Irving Kaplansky: *The direct product of rings.* (Abstract 48-11-304-*t*.)

16. Edward Kasner: *Differential equations of the type:* $y''' = Gy'' + Hy''^2$. (Abstract 48-9-277-*t*.)

17. Edward Kasner: *Differential equations of the type:* $y^{iv} = Ay'''^2 + By''' + C$. (Abstract 48-9-278-*t*.)

18. Edward Kasner: *The close packing of spheres.* (Abstract 49-1-72-*t*.)

19. Ervand Kogbetliantz: *Inequality for definite integrals.* Preliminary report. (Abstract 48-11-315-*t*.)

20. A. N. Lowan and H. E. Salzer: *Coefficients in formulae for numerical integration.* (Abstract 48-7-238-*t*.)

21. Szolem Mandelbrojt: *Quasi-analyticity and properties of flatness of entire functions.* (Abstract 48-11-318-*t*.)

22. J. F. Ritt: *Bézout's theorem and algebraic differential equations.* (Abstract 48-11-308-*t*.)

23. R. M. Robinson: *Bounded analytic functions.* (Abstract 48-11-320-*t*.)

24. J. A. Shohat: *An inequality for definite integrals.* Preliminary report. (Abstract 48-11-323-*t*.)

25. A. D. Wallace: *A characterization of unicoherence.* (Abstract 48-11-345-*t*.)

26. Y. C. Wong: *Some Einstein spaces with conformally separable fundamental tensors.* (Abstract 49-1-83-*t*.)

27. Jakob Levitski: *On the radical of a general ring.* (Abstract 49-1-7-*t*.)

T. R. HOLLCROFT,
Associate Secretary

BOOK REVIEWS

Sur les ensembles de distances des ensembles de points d'un espace Euclidien. (Mémoires de l'Université de Neuchatel, vol. 13.) By Sophie Piccard. L'Université de Neuchatel, 1939. 212 pp.

Let A denote a set of points on the real number-line R. The set of all number $|a-a'|$, for any a and a' in A, is the set of distances of the set A, and is denoted by $D(A)$.

A connection between A and $D(A)$ arises naturally in the following way. Let r be a real number, and let $x \to y = x+r$ be a translation of R carrying x into y. Let this translation carry A into $A(r)$. A number z will be in both A and $A(r)$ if and only if z is in A and also at a directed distance r from some point of A. Thus the intersection $A \cdot A(r)$ contains exactly one point for each distinct unordered pair (a, a') of elements of A such that $|a-a'| = |r|$. It is not surprising, then, to find many theorems based on the cardinal number of those r for which the cardinal number of $A \cdot A(r)$ is restricted in some way.

This particular connection is of largely auxiliary interest, however, for the main purpose of the present memoir is to examine for their own sake the relations between properties of A and $D(A)$. The results are so numerous and so detailed that we are obliged merely to cite a few typical items from each of the four chapters. Definitions and notation are as follows: A is congruent to B ($A \cong B$) when and only when a real number k can be found such that one of the transformations $x \to y = \pm x + k$ of R carries A into B. The complement of A in R is denoted by CA. The complement of $D(A)$ in the set $0 \leqq x < \infty$ is denoted by $D'(A)$ (reviewer's notation); since 0 is in $D(A)$ naturally, any member of $D'(A)$ is positive. If A and B are two sets, $D(A, B)$ is the set of all numbers $|a-b|$ for any a in A and b in B.

The most important material in the first chapter is a summary of general theorems, due mainly to Sierpinski, whose acknowledged influence appears clearly throughout the book. We learn, for instance, that, if A is open, an F_δ, a denumerable G_δ, Borel-measurable and with $A \cdot A(r)$ at most denumerable for each r, analytic, or of the second Baire category, then $D(A)$ is, respectively, a G_δ, an F_δ, a $G_{\delta\sigma}$, Borel-measurable, analytic, or of the second Baire category. A short proof is given of Steinhaus' important theorem that, if A has positive measure, then $D(A)$ contains an interval $0 \leqq x < k$ for some k. Finally, a partial answer is given to the perfectly natural question: Evidently $A \cong B$ implies $D(A) = D(B)$; when does $D(A) = D(B)$ imply $A \cong B$? If A and B are finite, it is necessary and sufficient to assume

that, for each r, $A \cdot A(r)$ and $B \cdot B(r)$ each contains at most one element; if A and B are infinite, this assumption is insufficient.

In the second chapter, the set $\Delta(A) = D'(A) \cdot D'(CA)$ is of great importance. If $\Delta(A)$ is empty, one of its factors is empty. $\Delta(A)$ is not empty when and only when $A \cong CA$. The second chapter investigates $D(A)$ mainly on this hypothesis, which entails $D(A) = D(CA)$ and $\Delta(A) = D'(A) = D'(CA)$. Under these circumstances, $A(\pm d) = CA$ (and $(CA)(\mp d) = A$) if and only if d is in $D'(A)$. Further, if $p > 0$ and $k_1, \cdots, k_p$ are any integers, $m_1, \cdots, m_p$ are any members of $D'(A)$, and $r = k_1 m_1 + \cdots + k_p m_p$, then $|r|$ is in $D'(A)$ if and only if the integer $k_1 + \cdots + k_p$ is odd. $D'(A)$ is a proper subset of that one of A and CA which does not contain zero. If $D(A)$ contains an interval, $D'(A)$ contains no decreasing sequence (and hence is denumerable). If $\Delta(A)$ contains two incommensurable numbers, then $\Delta(A)$ contains a decreasing sequence, and is everywhere dense on the interval $0 \leqq x < \infty$, while $A \cong CA$ is everywhere dense on R. Many other results deal with measurability and the Baire categories.

The next chapter is devoted to perfect sets. If A is closed and bounded, or dense-in-itself, or perfect and bounded, then $D(A)$ is, respectively, closed, dense-in-itself, or perfect. A perfect set P is said to be of the first kind (Mirimanoff) if it arises from an interval $a \leqq x \leqq b$ by a process generalized in a specific way from the construction of the Cantor ternary set. If P is of the first kind, $D(P)$ is the interval $0 \leqq x \leqq b - a$. If P and Q are of the first kind, $D(P, Q)$ has positive Lebesgue-measure mes $D(P, Q)$. There exist bounded perfect sets P and Q such that mes $D(P) = 0$ and mes $D(Q) = 0$, while mes $D(P, Q) > 0$; there exist bounded perfect sets P and Q such that mes $D(P) > 0$ and mes $D(Q) > 0$, while mes $D(P, Q) = 0$. If mes $D(A) = 0$, then mes $D(A, A(r)) = 0$ for any real r. The remainder of this chapter contains a plethora of specific information on $D(A)$ and $D(A, B)$, where A and B consist of those numbers expressible in special ways in systems of notation with assigned bases. Presumably as an example, the base 4 receives extraordinarily searching attention.

The final chapter studies conditions on a subset D of the interval $0 \leqq x < \infty$ in order that there may exist an A for which $D = D(A)$. If D is finite, it is necessary and sufficient that some subset of D, taken in increasing order, shall form a sequence S such that D coincides with the sums of the gapless subsequences of S. (It may be necessary to repeat elements of D when forming S. Example: $A = \{0, a, 2a\} = D(A)$ with $a > 0$, $S = 0, a, a$.) Results which increase rapidly in complication are obtained directly when D has not more

than six elements, or when $A \cdot A(r)$ or $D \cdot D(r)$ are suitably restricted. With conditions on A, the criterion for finite D generalizes to infinite D. If $D(A)$ and (hence) A are denumerably (nondenumerably) infinite, then $D(A) \cdot (D(A))(r)$ is necessarily denumerably (nondenumerably) infinite for denumerably (nondenumerably) many r's. If D' is of measure zero or of the first Baire category, then the continuum-hypothesis implies the existence of an A for which $D = D(A)$. Finally, many properties of D are enumerated which exclude the existence of the required A.

Aside from occasional use of transfinite induction and the continuum-hypothesis, the proofs may fairly be called elementary. On the other hand, they are often quite intricate, simple results requiring the examination of a myriad cases. Pushed with patient energy, this study has yielded a remarkable amount of detailed information, from which one may form a definite idea of the difficulties and rewards to be met in the directions here pursued. One also finds general results of great interest, which do not claim to be complete. Many problems are explicitly proposed, and many others at once suggest themselves. For this solid progress we are very substantially indebted to the author, who doubtless shares the hope that a still deeper analysis—possibly along somewhat different lines—may presently yield a more satisfying theory.

Remarks: 1. Considerable data are given on sets A in higher dimensional spaces, but such results have been ignored here for the sake of brevity.

2. Though the argument is apparently not vitiated by its omission in the text (for reasons which will not escape a reader), the reviewer feels compelled to record here the fact that the property P (p. 54) is not invariant under translation, is accordingly peculiar to the position of a set on the line, and may not be shared by a set congruent to a set which has it.

3. One should take care, on p. 39, line 19, to read "$D(A)+D(CA)$" instead of "$D(A)$" (see Property 2, p. 46).

4. This book regrettably upholds the secretive tradition under which the index is omitted.

F. A. FICKEN

Tables of probability functions, Vol. II. New York, Work Projects Administration, 1942. 21+344 pp. $2.00.

The first volume of *Probability functions* appeared in 1941; it was reviewed in Bull. Amer. Math. Soc. vol. 48 (1942) p. 201. The present

volume tabulates the functions

$$I=\frac{1}{(2\pi)^{1/2}}e^{-x^2/2}, \qquad H=\frac{1}{(2\pi)^{1/2}}\int_{-x}^{x}e^{-y^2/2}dy$$

from 0.0000 to 1.0000 at intervals of 0.0001 to fifteen places of decimals, and from 1.000 to 8.285 at intervals of 0.001 to fifteen places. Tables II include the functions I and $1-H$ from 6.00 to 10.00 at intervals of 0.01 to six places of decimals. The bibliographical data provided give valuable information concerning uses of the tables. The extensive and systematic tests for accuracy are as in the earlier volumes of the series.

VIRGIL SNYDER

Tables of sine and cosine integrals for arguments from 10 *to* 100. New York, Work Projects Administration, 1942. 185 pp. $2.00.

The present volume is a continuation of *Tables of sine, cosine and exponential integrals*, vols. I and II which appeared in 1940 and were reviewed in Bull. Amer. Math. Soc. vol. 47 (1941) pp. 677–678, except that exponential integrals are not included over the new interval. New features in the present volume are graphs of $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ and a bibliography of applications as well as of tables and of reference texts.

The Tables themselves are to ten places, at intervals of 0.01 from 10 to 100, arranged as in the earlier volumes. Then follow $n\pi/2$ to fifteen places, n ranging by integers from 1 to 100, and $p(1-p)$ and $p(1-p^2)$, each to six places.

The Tables are reproduced by the photo-offset process as were the previous volumes. Such use has been made in the checks and controls described in the Introduction as to secure a very high degree of accuracy in the results.

VIRGIL SNYDER

NOTES

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The Kosciuszko Foundation announces that for May 23, 1943, it is planning a scientific tribute in commemoration of the four hundredth anniversary of the death of Nicolas Copernicus, the great Polish astronomer. This is also the four hundredth anniversary of the publication of his work, *De revolutionibus orbium coelestium.*

M. Krylov will head a special committee of the Soviet Academy of Sciences to arrange for celebration of the tercentenary of Newton's birth.

Applications for the Benjamin Peirce instructorships at Harvard University for the academic year 1942–1943 should be sent to the chairman of the department of mathematics. Candidates should have received the Ph.D. degree or have had equivalent training.

Haverford College has conferred the doctorate of science upon Professor H. V. Gummere, who retired from its staff in June, 1942.

Assistant Professor E. F. Beckenbach of the University of Michigan has been appointed to an associate professorship at the University of Texas.

Professor R. E. Bruce of Boston University has been given the title professor emeritus.

Assistant Professor P. N. Carpenter of Grove City College has been promoted to an associate professorship.

Professor Helen E. Clarkson of Jacksonville College, Jacksonville, Texas, has been appointed to an assistant professorship of physics at Hardin-Simmons University.

Professor L. W. Cohen of the University of Kentucky has been appointed visiting lecturer at the University of Wisconsin for this academic year.

Mr. H. W. Eves of the mathematics staff of the Tennessee Valley Authority has been appointed to an assistant professorship of applied mathematics at Syracuse University.

Dr. Mary C. Graustein of Radcliffe College has been appointed to an assistant professorship at Oberlin College.

Professor Emeritus J. G. Hardy of Williams College will be visiting professor of mathematics at Reed College during the current academic year.

Dr. H. A. Jordan of the Catholic University of America has been appointed to an assistant professorship at the College of William and Mary.

Assistant Professor F. H. Miller of the Cooper Union School of Engineering has been promoted to an associate professorship.

Dr. J. W. Odle has been appointed to an assistant professorship at Pennsylvania State College.

Dr. E. W. Paxson of Wayne University has been promoted to an assistant professorship.

Dr. Grace S. Quinn has been appointed associate in mathematics at George Washington University.

Assistant Professor W. J. Robinson of Washington College, Chestertown, Maryland, has been appointed to an assistant professorship at Marshall College, Huntington, West Virginia.

Professor A. J. Smith of Susquehanna College has been appointed to an assistant professorship at the College of William and Mary.

Assistant Professor F. H. Steen of Georgia School of Technology has been appointed to an assistant professorship at Allegheny College.

Assistant Professor N. E. Steenrod of the University of Chicago has been appointed to an assistant professorship at the University of Michigan.

Professor H. P. Thielman of the College of St. Thomas, St. Paul, Minnesota, has been appointed to an assistant professorship at Iowa State College.

Dr. G. R. Thurman of the University of Missouri has been appointed to an assistant professorship at Tusculum College, Greenville, Tennessee.

Dr. W. R. Van Voorhis of Fenn College ha. been promoted to an assistant professorship.

Dr. P. A. White has been appointed to an assistant professorship at Louisiana State University.

Associate Professor J. H. Zant of Oklahoma Agricultural and Mechanical College has been promoted to a professorship.

The following appointments to instructorships have been announced: Cornell University: Mr. R. L. Beinert, Mr. C. D. Firestone, Mr. R. L. Hull, Mr. A. R. Turquette; Grays Harbor Junior College, Aberdeen, Washington: Miss Ruth E. Porter; Illinois Institute of Technology: Dr. W. B. Caton; Louisiana State University: Mr. J. C. Stewart; University of Maryland: Mr. H. F. Gingerich, Dr. Margaret S. Matchett; Massachusetts Institute of Technology: Dr. W. F. Whitmore; Michigan State College: Dr. L. V. Toralballa, Miss Elaine

Van Aken, Mr. K. C. Walters; New York University: Dr. Wolfgang Wasow; Queens College: Dr. Deborah M. Maria; University of Wyoming: Dr. Hans Samelson.

A recent message received through the American Red Cross announces the death in Germany on July 5, 1942, of Professor Oskar Bolza at the age of 85 years. He had been a member of the Society since 1891.

Nature reports the death of the English mathematician D. B. Mair.

Professor A. D. Campbell of Syracuse University died September 23, 1942. He had been a member of the Society since 1920.

Professor C. N. Haskins of Dartmouth College died November 14, 1942, at the age of 68 years. He had been a member of the Society since 1913.

Professor Emeritus D. A. Lehman of Goshen College died September 8, 1942, at the age of 82 years.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

ALGEBRA AND THEORY OF NUMBERS

1. I. S. Cohen: *Some theorems on local rings.*

Let $\mathfrak{R}$ be a p-series ring (as defined by Krull, J. Reine Angew. Math. vol. 179 (1938) p. 205); let n be its dimension. It is shown that if $\mathfrak{a}$ is an ideal in $\mathfrak{R}$ having a basis of r elements and of dimension at most $n-r$, than $\mathfrak{a}$ is unmixed, of dimension $n-r$. This is a generalization of a well known theorem of Macaulay. If $\mathfrak{R}$ is complete and if $\mathfrak{S}$ is a p-series ring integrally dependent on $\mathfrak{R}$, then the rank of $\mathfrak{S}$ over $\mathfrak{R}$ is equal to the ramification order of $\mathfrak{S}$ with respect to $\mathfrak{R}$ (defined as the length of the primary ideal obtained by extending to $\mathfrak{S}$ the maximal ideal of $\mathfrak{R}$) multiplied by the degree of the residue field of $\mathfrak{S}$ over that of $\mathfrak{R}$. These results depend on the following theorem concerning the structure of a complete p-series ring: If the characteristics of $\mathfrak{R}$ and its residue field are the same, then $\mathfrak{R}$ is a power series ring over this field; if the characteristics are different, then under a simple additional hypothesis, $\mathfrak{R}$ is a power series ring over a complete discrete valuation ring. It is also shown that every complete local ring is a homomorphic map of a ring of one of these two types. (Received November 24, 1942.)

2. Franklin Haimo: *Periodic functions on algebraic systems.*

A single-valued function F over a group G is said to have a period p if $F(xpy) = F(xy)$ for every x and y in G. The periods of F form a normal subgroup of G. All single-valued functions over G with range in a class C are at least trivially periodic and are partitioned into classes each containing one and only one distinct homomorphism of G. Single-valued functions over a quasi-field H may have periods which are both additive and multiplicative. Such periods form a normal subgroup of the multiplicative group of H. Single-valued functions over lattices may have join-periods and meet-periods. Meet-periods form a join-ideal and dually; while non-constant single-valued functions over Boolean algebras have no members which are both join- and meet-periods. (Received November 23, 1942.)

3. P. R. Halmos: *On automorphisms of compact groups.* I.

If T is a continuous automorphism of a compact abelian group G then, because of the uniqueness of Haar measure, T is a measure preserving transformation. T is ergodic (in fact strongly mixing) if and only if the adjoint automorphism T^* of the character group G^* has no finite orbits. If an automorphism (such as T^*) of a discrete abelian group has no finite orbits then it has an infinite number of orbits. It follows

easily that the spectral type of the unitary operator induced by an ergodic T on $L_2(G)$ depends only on the cardinal number of G^*. In case G is a finite-dimensional toral group (so that T is defined by a unimodular matrix) a simple condition on the coefficients of T is equivalent to ergodicity. This last remark puts in evidence many new and simple examples of analytic mixing transformations. A purely measure theoretic invariant of measure preserving transformations is proposed (along the lines of the characteristic equation theory of matrices); it is suggested that this invariant will serve to distinguish between transformations lumped together by the cruder spectral invariants. (Received November 17, 1942.)

4. Gerald Harrison: *The structure of algebraic moduls.*

A 1-1 correspondence between the submoduls of an algebraic modul of degree n and the set of all $n \times m$ matrices whose elements are rational integers may be set up if equivalence of matrices is defined in the usual manner. Nonsingular matrices whose elements are rational integers have multiplicative decompositions as the product of prime matrices, that is, matrices whose norms are primes. The number of such decompositions which a matrix has may be described in terms of the modular lattice which the set of all nonsingular matrices forms. This lattice has a direct product decomposition which is at the same time multiplicative, that is, if $a=(a_1, a_2, \cdots)$ and $b=(b_1, b_2, \cdots)$ then $a \cup b=(a_1 \cup b_1, a_2 \cup b_2, \cdots)$, $a \cap b=(a_1 \cap b_1, a_2 \cap b_2, \cdots)$, and $ab=(a_1b_1, a_2b_2, \cdots)$, where the multiplication referred to is that of algebraic moduls. (Received November 23, 1942.)

5. Olaf Helmer: *An extension of the elementary divisor theorem.*

A Prüfer ring is a domain of integrity in which all ideals possessing a finite basis are principal ideals. The elementary divisor theorem is known to hold in principal ideal rings as well as in rings with Euclidean algorithm; whether it will ever be possible to extend the theorem to Prüfer rings is questionable. In this paper it will be proved for a subclass of all Prüfer rings, to be called adequate rings, which more than comprise the class of rings hitherto known to be covered by the theorem. An adequate ring is defined as a Prüfer ring in which every nonzero element a can, for any element b, be split into two factors: $a=a_1 \cdot a_2$, such that a_1 is a largest factor of a prime to b: $(a_1, b)=1$, largest in the sense that no factor a_3 of a_2 is prime to b: $(a_3, b) \neq 1$. The proof proceeds along familiar lines, after the following lemma has been proved: Let $M=(a_{ij})$, $(i=1, \cdots, r; j=1, \cdots, n)$ be a matrix with coefficients in an adequate ring R, let $1<r \leq n$, rank $M=r$, and $(a_{11}, a_{12}, \cdots, a_{rn})=d$; then there exist $t_1, t_2, \cdots, t_{r-1}$ in R such that $(A_1, A_2, \cdots, A_n)=d$, where $A_j=a_{1j}t_1+\cdots+a_{r-1,j}t_{r-1}+a_{rj}$. (Received November 23, 1942.)

6. Jakob Levitzki: *On semi-nilpotent ideals.*

A ring T is called semi-nilpotent (in short: s.n.) if each subring of T which is generated by a finite subset of T is nilpotent; otherwise T is called semi-regular. Thus the sum $\overline{N}$ of all nilpotent right (left) ideals of any ring S is s.n. In a previous note the author has proved that each ring S possesses a uniquely determined maximal two sided s.n. ideal N which contains also all one-sided s.n. ideals of the ring. In the present note it is shown that if S is a ring with minimal condition for two-sided s.n. ideals, then N is nilpotent, and hence $N=\overline{N}$. Applied to semi-primary rings, this theorem yields certain generalizations of recent results due to C. Hopkins (Duke

Math. J. vol. 4 (1938) pp. 664–667), J. Levitzki (Compositio Math. vol. 7 (1939) pp. 214–222) and K. Asano (Proc. Imp. Acad. Tokyo vol. 15 (1939) pp. 288–291). (Received November 23, 1942.)

7. Jakob Levitski: *On the radical of a general ring.*

In the present note a definition of the radical is suggested which retains its significance also in the general case. This definition is based on the notion of s.n. (semi-nilpotent) ideal (its counterpart is the semi-regular ideal) which is defined as follows: R is a s.n. ideal if each finite set of elements in R generates a nilpotent ring. Each nilpotent ideal is s.n., and each s.n. ideal is a nil-ideal. The sum N of all right s.n. ideals is a s.n. two-sided ideal, which is called the radical. The radical N of any ring S contains all one-sided (and two-sided) s.n. ideals, and the radical of S/N is zero. These theorems lose their validity for general rings if the radical is defined (as has been hitherto the case) by replacing s.n. ideals either by nilpotent ideals (specialized radical) or nil-ideals (generalized radical). The radical contains the specialized radical, and is a subset of the generalized radical (in case the latter exists). The results described above are applied to primary rings. (Received October 5, 1942.)

8. Saunders MacLane and B. A. Lengyel: *Integral invariants of a tensor under the symmetry operations of the 32 crystal classes.* Preliminary report.

The theory of elastic and plastic deformations of an isotropic body makes use of the invariants of the stress tensor under the group of all orthogonal transformations. To extend this theory to crystalline media it is necessary to know the invariants of the finite groups of transformations characteristic of the crystal classes. The effect of the various symmetry operations on the components of a tensor was studied and an integrity basis was set up for the invariant polynomials for each of the 32 classes. The method of finding the integrity bases is elementary and consists of reducing the problem to that of the vector invariants of the symmetric and alternating groups, respectively. (Received November 24, 1942.)

9. C. J. Nesbitt and W. M. Scott: *Some remarks on algebras over an algebraically closed field.*

In this paper are examined some aspects of an algebra A which may have a radical and whose coefficient field is algebraically closed. A main concept is that of basic algebra. The basic algebra of A is a semi-primary subalgebra which for the algebra A plays a role in some respects analogous to that of division algebras in the theory of simple algebras. Related to the basic algebra are the Cartan basis systems and systems of elementary modules (W. M. Scott, Ann. of Math. (2) vol. 43 (1942) pp. 147–160). An algebra B is said to be similar to A if the basic algebra of B is isomorphic to that of A. Similar algebras form a class, and have corresponding representations. The commutator algebras of matrix representations of A are analyzed. The linear symmetric functions of A (abstract 44-11-452) are determined. The regular representations of A are written in terms of the elementary modules. (Received October 26, 1942.)

10. Ivan Niven: *An unsolved case of the Waring problem.*

The Dickson-Pillai-Vinogradow solution of Waring's problem (L. E. Dickson,

Amer. J. Math. vol. 58 (1936) p. 535) leaves unsolved the case in which r, the remainder upon division of 3^n by 2^n, equals 2^n-q-2, q being the quotient. The present paper shows that in this case $g(n)$, the minimum number of positive or zero nth powers necessary to express every positive integer, has the "ideal" value 2^n+q-2. The method used is Dickson's, sharpened somewhat. (Received October 14, 1942.)

11. Ivan Niven: *The Pell equation in quadratic fields.*

In a previous report (*Quadratic Diophantine equations in the rational and quadratic fields*, Trans. Amer. Math. Soc. vol. 52 (1942) p. 2, Theorem 4) the author gave necessary and sufficient conditions that the equation $\xi^2-\gamma\eta^2=1$ have an infinite number of integral solutions ξ, η in any quadratic field, γ being a given integer of the field. It is shown here that if the equation has an infinitude of solutions, they can be obtained from one least solution (as in the case of the Pell equation in the rational field) if and only if γ is not a totally positive non-square integer of a real quadratic field. In case γ has this property it is shown that the equation has infinitely many solutions with ξ and η bounded. (Received October 16, 1942.)

12. Rufus Oldenburger: *The characteristic of a sum of quadratic forms.*

It is proved that the Loewy characteristic of a sum of quadratic forms is determined by properties of adjoint forms, and nonzero components in characteristic splittings of quadratic forms. (Received November 21, 1942.)

13. Edward Rosenthall: *Diophantine equations in arbitrary algebraic number fields.*

The complete solution in integers $XY=ZW$ in any algebraic integral domain is given by $eX=US$, $eY=VT$, $eZ=UT$, $eW=VS$ where e takes only the finite set of rational integral values, each equal to the norm of a representative ideal from each class, and U, S, V, T, are arbitrary integers of the field. Complete integer solutions of other multiplicative equations are deduced and in particular from one of these equations all sets of rational integers satisfying $ax^2+by^2=z^2$ are obtained in terms of arbitrary integral parameters subject to a congruential condition modulo e. (Received November 20, 1942.)

14. Edward Rosenthall: *Diophantine equations in biquadratic fields.*

Explicit complete rational integer solutions are obtained for some diophantine equations reducible in certain so-called special Dirichlet biquadratic fields. The equation $N(X)=N(Y)$ is solved completely in rational integers, where $N(A)$ denotes the normal of the biquadratic integer A; in particular this equation in $Ra[2^{1/2}+i]$ yields the complete rational integer solution of $x^4+y^4=N(s+it+2^{1/2}u+i2^{1/2}v)$ with at most one linear relationship connecting the coordinates s, t, u, v. A multiplicative equation in $Ra[3^{1/2}+i]$ gives the complete rational integer solution of $x^3+y^3=u^2+v^2$. The equation $x^3+y^3=u^3+v^3$ is also solved completely in Gaussian integers. (Received November 20, 1942.)

15. T. L. Wade: *Euclidean concomitants of the triangle.*

The results of the writer (Bull. Amer. Math. Soc. vol. 47 (1941) pp. 475–478 and

vol. 48 (1942) pp. 589–601) which relate tensor algebra and invariant theory are used in the consideration of the elementary problem of the three-line configuration. (Received November 23, 1942.)

16. T. L. Wade: *On conjugate tensors.*

How a contravariant (skew-symmetric) tensor V of order $n-p$ may be associated with a covariant skew-symmetric tensor U of order p, in an n-dimensional coordinate system, is well known (see Veblen and von Neumann, *Geometry of complex domains*). This standard association holds only when U is skew-symmetric. The purpose of this note is to show how a contravariant tensor V, of defined order and symmetry, can be associated with the covariant tensor U, where U is of any type $[\alpha]$ of symmetry. (Received November 23, 1942.)

17. T. L. Wade: *On the factorization of rank tensors.*

Let $C_{(\rho)}^{(\omega)}=D_{(\rho)}^{(\omega)}+E_{(\rho)}^{(\omega)}$, where $D_{(\rho)}^{(\omega)}$ and $E_{(\rho)}^{(\omega)}$ are mutually orthogonal idempotent numerical tensors. Expressions for the contravariant and covariant factors of the rank tensor (see Amer. J. Math. vol. 64 (1942) pp. 725–752) of $C_{(\rho)}^{(\omega)}$ in terms of like factors of the rank tensors of $D_{(\rho)}^{(\omega)}$ and $E_{(\rho)}^{(\omega)}$ are established in this paper. (Received November 23, 1942.)

18. T. L. Wade and R. H. Bruck: *Types of symmetries.*

This paper considers some aspects of symmetries with tensorial significance which are believed not to have appeared in the literature. (Received November 23, 1942.)

19. André Weil: *Differentiation in algebraic number-fields.*

Analogies with function-fields have long ago led E. Noether and others to the conjecture that the theory of the different in number-fields can be built upon some arithmetical analogue of differentiation. This is now done, by defining a derivation modulo an ideal $\mathfrak{a}$ in a number-field as an operator D with the following properties: (a) D maps the ring $\mathfrak{o}$ of integers in the field into the ring $\mathfrak{o}/\mathfrak{a}$; (b) $D(\alpha+\beta)=D\alpha+D\beta$; (c) if $\bar{\alpha}$, $\bar{\beta}$ are the classes of α, β mod $\mathfrak{a}$, then $D(\alpha\beta)=\bar{\alpha}\cdot D\beta+\bar{\beta}\cdot D\alpha$; D is essential if there is α in $\mathfrak{o}$, such that $D\alpha$ is not a zero-divisor in $\mathfrak{o}/\mathfrak{a}$. The different is then the least common multiple of all ideals modulo which there exists an essential derivation. This is easily extended to the relative different, to p-adic fields, and so on. (Received November 9, 1942.)

20. Alexander Wundheiler: *An algebraic definition of affine space.*

A simple set of axioms for affine geometry based on one operation $C=hAB$, where h is a real number, A, B points and C the point collinear with A and B, and such that $CA/CB=h$, is given. There are essentially five axioms, only one of them involving more than two (namely, three) points. An "affine calculus," which permits the writing of every affine theorem as an implication between formulas, arises from the mentioned operation. (Received November 20, 1942.)

Analysis

21. R. P. Agnew: *Euler transformations.*

Let $E(r)$ denote the Euler transformation $\sigma_n=\sum_{k=0}^{n}C_{nk}r^k(1-r)^{n-k}s_k$ by means of

which a sequence s_n is summable to σ if $\sigma_n \to \sigma$ as $n \to \infty$. The fundamental properties of the transformations $E(r)$ are developed for the general case in which r is complex. The family of transformations $E(r)$ for which $r \neq 0$ is a consistent family even though $E(r)$ is regular only when r is real and $0 < r \leq 1$. Relations between the methods $E(r)$ and other methods of summability are established. The members of a family of series-to-series transformations are shown to be equivalent to the transformations $E(r)$ when the parameters are suitably restricted. The open sets in which power series are summable are characterized. If $r_1, r_2, \cdots$ is a sequence of real numbers for which $r_n \to 0$ and $nr_n \to \infty$, then the transformation $\sigma_n = \sum_{k=0}^{n} C_{nk} r_n^k (1 - r_n)^{n-k} s_k$ includes $E(r)$ for each $r > 0$. (Received October 13, 1942.)

22. R. P. Agnew: *On sequences with vanishing even or odd differences.*

Let $x_0, x_1, x_2, \cdots$ be a sequence of real or complex numbers, and let $d_n = \sum_{k=0}^{n} (-1)^k C_{n,k} x_k$, $n = 0, 1, 2, \cdots$, denote its sequence of differences. By use of a theorem on Euler methods of summability, a short proof of the two following theorems is obtained. If x_n if bounded and $d_n = 0$ when n is even, then $x_n = 0$ for each n. If x_n is bounded and $d_n = 0$ when n is odd, then $x_n = x_0$ for each n. (Received October 13, 1942.)

23. I. A. Barnett: *A note on skew-symmetric kernels.*

It is proved in this note that a necessary and sufficient condition that $\Gamma(x, y; \lambda)$ be the resolvent kernel of a skew-symmetric kernel is that $\Gamma(x, y; \lambda) + \Gamma(y, x; \lambda) + 2\lambda \int_a^b \Gamma(x, t; \lambda)\, \Gamma(y, t; \lambda)\, dt$ be identically zero in x, y and λ. This is the analogue of the fact that the adjoint $C(\lambda)$ of $A - \lambda I$ satisfies the equation $\det(A - \lambda I)[C(\lambda) + C'(\lambda)] + 2\lambda C(\lambda) C'(\lambda) = 0$, where A is a skew-symmetric matrix, and $C'(\lambda)$ denotes the transpose of $C(\lambda)$ (cf. Amer. Math. Monthly vol. 49 (1942) pp. 169–170). As a consequence of the first relation, it follows that a solution of the quadratic integral equation $\phi(x, x) = \lambda \int_a^b \phi^2(x, t) dt$ is $\phi(x, y) = -\Gamma(x, y; \lambda)$ where Γ is the resolvent of an arbitrary skew-symmetric kernel. (Received November 20, 1942.)

24. E. F. Beckenbach: *The stronger form of Cauchy's integral theorem.*

According to the stronger form of Cauchy's integral theorem, if $f(z)$ is holomorphic on the interior D of a simply closed rectifiable curve C, and continuous on $D + C$, then $\int_C f(z) dz = 0$. A brief and simple though not elementary proof is given, based on considerations of conformal mapping, the length of level curves of the Green's function, and a simple property of a Stieltjes integral. (Received November 23, 1942.)

25. Henry Blumberg: *On arbitrary point transformations.*

The transformations τ considered are correspondences $\tau: y = f(x)$ which associate with each point $x: (x_1, x_2, \cdots, x_m)$ of euclidean m-space S_m a point $y: (y_1, y_2, \cdots, y_n)$ of euclidean n-space S_n. No other conditions are imposed on τ except "one-valuedness," which, too, is not an essential one. A series of theorems are proved, of descriptive and matric character, for the structure of τ. The principal results are of the latter sort, and relate to the notion of "salient point," defined as a point of non-infinitesimal approach—in a certain readily suggested sense—of the "graph" of τ in

S_{m+n}, a euclidean space of $m+n$ dimensions. Theorems are obtained, among others on the distribution, closure properties, and measurability—in a certain sense—of the set of salient points. Finally, necessary and sufficient conditions are established characterizing such a set. (Received November 19, 1942.)

26. R. F. Clippinger: *General remarks about the set of products of positive powers of n-by-n matrices and the associated manifold.* Preliminary report.

Let P be the set of products of arbitrary positive powers of the exponentials of m given n-by-n matrices, A_i, of real numbers. Let M be the corresponding manifold of points of n^2 Euclidean space. Let $\overline{P}$ and $\overline{M}$ be the closure of P and the corresponding manifold. P is a groupoid. M is connected and r-dimensional if r is the number of linearly independent matrices B_j among A_i and their alternants. If C_k is an arbitrary linear combination of the B_j, there exists a number l such that any matrix of P is a product of at most l matrices $\exp C_k$. $\overline{P}$ contains all matrices of the form $\exp A$ if A is any linear combination with positive coefficients of A_i. $\overline{P}$ is identical with the closure of the set of particular values of solutions of the differential equation $dY/dt = YB$, $Y(a) = 1$ where B is a linear combination of A_i with coefficients which are arbitrary non-negative functions of class C^∞. (Received November 21, 1942.)

27. R. F. Clippinger: *Matrix products of matrix powers.* Preliminary report.

If A and B are 2-by-2 matrices with constant elements a, b, c, d, e, f, g, h and distinct characteristic roots such that A, B, (AB) and $(A(AB))$ are linearly independent and $[(bg+cf-ah-de)(e+h)+2(eh-fg)(a+d)][(bg+fc-ah-de)(a+d)+2(ad-bc)(e+h)]$ is negative, and if $\alpha, \beta, \gamma, \delta, \alpha_1, \beta_1, \alpha_2, \beta_2, \cdots$ are arbitrary non-negative numbers, then the closure of the set of matrices $\exp \alpha_1 A \exp \beta_1 B \exp \alpha_2 A \cdot \exp \beta_2 B \cdots$ is identical with the closure of the set of matrices $\exp \alpha A \exp \beta B \exp \gamma A \cdot \exp \delta B$. (Received November 21, 1942.)

28. R. F. Clippinger: *Mean value theorems for a certain linear matrix differential equation.* Preliminary report.

Let A and B be two, n-by-n matrices, with distinct characteristic roots, which verify the condition $AB-BA=\lambda A+\mu B$, $\mu \neq 0$. If $\rho(t)$ and $\sigma(t)$ are arbitrary, non-negative, bounded measurable functions of t on the interval $a \leqq t \leqq b$, and if $Y(b)$ is the particular value for $t=b$ of the solution of the matrix differential equation $dY(t)/dt = Y(t)[\rho(t)A+\sigma(t)B]$, $Y(a)=1$, then there exist non-negative numbers $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ such that, without changing $Y(b)$, $\rho(t)$ and $\mu(t)$ may be replaced by: (a) α and β on $a \leqq t \leqq b$; or, (unless $\mu>0$ and $\lambda>0$), (b) γ and 0 on $a \leqq t \leqq (a+b)/2$; 0 and δ on $(a+b)/2 \leqq t \leqq b$; or, (unless $\mu<0$ and $\lambda<0$), (c) 0 and δ on $a \leqq t \leqq (a+b)/2$; γ and 0 on $(a+b)/2 \leqq t \leqq b$. (Received November 21, 1942.)

29. Max Coral: *Solution of quasi-linear partial differential equations through a characteristic initial curve.*

A new method is presented of passing a solution of a quasi-linear first order partial

differential equation in two independent variables through an initial curve C which is characteristic for the equation. It is shown that C yields a solution S of the equations of variation of the characteristic differential equations. If T is any solution of these equations of variation which is independent of S, then a one-parameter family of characteristic curves imbedding C can be found, whose variation along C is T. These characteristic curves envelop the desired solution. The present method has the advantage that the solution thus found contains the entire extent of the initial curve. (Received October 30, 1942.)

30. G. M. Ewing: *Minimizing an integral on a class of continuous curves.*

Hahn published an example in 1925 of a positive semidefinite positive quasi-regular variation problem which admits no rectifiable minimizing curve. Menger has obtained several existence theorems for problems in very general spaces which apply to the example of Hahn. In the present paper attention is confined to variation problems in euclidean spaces. The integral J is defined for a continuous curve C as g.l.b. lim inf $J(C_n)$, the greatest lower bound being on the class of sequences C_n of rectifiable continuous curves converging to C. Using δ-length based on any distance $\delta(p, q)$ satisfying the axioms of a metric space, a set of admissible curves of bounded δ-length is compact. The method involves finding a suitable metric δ and determining hypotheses under which the problem admits a minimizing sequence of rectifiable curves of bounded δ-length. Existence theorems based on several choices of δ and related lower-semicontinuity theorems are discussed. (Received October 14, 1942.)

31. N. A. Hall: *Confluence of the basic hypergeometric series.*

The general properties of the basic hypergeometric series are reviewed with special attention being given to the introduction of a satisfactory comprehensive notation and to the establishment of criteria for identifying analogues of theorems concerning generalized hypergeometric series. In particular two analogues of the confluence of generalized hypergeometric series are discussed and applied to several identities appearing in mathematical literature. Examples are given relating these confluent basic series to identities of modular functions. It is shown that the whole field of theta function, mock-theta function, modular function, and Raman Ramanujan identities may probably be expressed very elegantly and completely in terms of the basic hypergeometric series and their established relations. (Received November 19, 1942.)

32. Einar Hille and Max Zorn: *Open additive semi-groups of complex numbers.*

The main object of this paper is to determine the open, connected and additive semi-groups of the complex number plane. These sets are important as parameter manifolds of semi-groups of linear transformations which have been studied in detail by Hille. It is shown that the open semi-groups of the plane depend upon the upper-semi-continuous solutions of the inequality $f(x+y) \leqq f(x)+f(y)$, the function f being defined for all real or all positive (negative) numbers x. These sets are automatically connected and simply connected. They are also shown to be the maximal domain of existence for suitable 1-parameter semi-groups of linear transformations in appropriate Banach spaces. Many of the results about the parameter manifolds permit considerable generalizations. (Received November 28, 1942.)

33. H. K. Hughes and Cleota G. Fry: *Asymptotic developments of certain integral functions.*

The authors obtain asymptotic developments valid for value of z of large modulus, of the function $f_\alpha(z)=\sum_{n=0}^{\infty}g(n)z^n$, where $g(n)$ is of the form $h(n)\{\Gamma(\alpha n+p)\}^{-1}$; $\alpha>0$; p is any constant, real or complex. It is assumed that the function $g(w)$, where $w=x+iy$, is single-valued and analytic in the finite w-plane, and that $h(w)$ is developable in a factorial series of the form $\sum_{n=0}^{\infty}a_n\Gamma(\alpha w+p)/\Gamma(\alpha w+p+n)$. The paper may be regarded as a generalization of results already obtained by W. B. Ford, and appearing in chap. 6 of his book entitled *Asymptotic developments of functions defined by Maclaurin series* (Michigan Science Series, vol. 11, Ann Arbor, 1936). The methods employed in the paper are essentially those employed by Ford. (Received November 20, 1942.)

34. Glenn James: *On equivalence of methods of summing divergent series.*

This paper investigates the uniqueness of sums given by the general, regular, definition $\lim_{n\to\infty}\sum_1^n s_i\alpha_i=S$, where the α_i's are positive or zero and converge uniformly to zero and $\lim_{n\to\infty}\sum_1^n\alpha_i=1$. It is shown that S is generally arbitrary. A fourth restriction is then placed upon the α_i's which, at least for most ordinary summable series, makes S unique, and enables one actually to determine S. This restriction is that the limit of the quotient of the number of changes in monotony in the sequence by n, as n increases, shall be zero. (Received October 31, 1942.)

35. H. F. S. Jonah and A. H. Smith: *Zero order summability of the series conjugate to the derived Fourier series.*

This paper proves that a theorem of Takahashi (Jap. J. Math. vol. 10 (1933) pp. 127–132) concerning the Cesàro summability of the series conjugate to the derived Fourier series remains true if the Cesàro method of summation is replaced by the zero order method of Bosanquet-Linfoot (J. London Math. Soc. vol. 6 (1931) pp. 117–126), which is weaker than the Cesàro method of any order $\alpha>0$. (Received October 29, 1942.)

36. Knox Millsaps: *Characterization of the abstract exponential function.*

After defining the abstract exponential function over normed rings, it is possible to characterize this function as the solution of a first order general differential system. The validity of the theorem is established by the use of two theorems of Kerner on the existence of a solution for such equations and the symmetry of second order Fréchet differentials. (Received October 31, 1942.)

37. C. N. Moore: *On the relationship between Nörlund means of a certain type.*

The purpose of this paper is to obtain relationships of inclusion between Nörlund means defined in terms of the coefficients of the power series developments of functions analytic within the unit circle and having a singularity of algebraic-logarithmic type at the point $z=1$. The inclusion relationship is found to depend on the relative

order of infinity of the functions involved in the neighborhood of the singular point. (Received November 19, 1942.)

38. G. B. Price: *Cauchy-Stieltjes and Riemann-Stieltjes integrals.*

This note treats the equivalence of the Riemann-Stieltjes and Cauchy-Stieltjes integrals (abbreviated RS and CS integrals) and conditions for the existence and equality of the latter. The RS and the left and right CS integrals are defined as limits of the sums $\sum_1^n f(\xi_i)[g(x_i)-g(x_{i-1})]$, $x_{i-1}\leqq\xi_i\leqq x_i$, $\sum_1^n f(x_{i-1})[g(x_i)-g(x_{i-1})]$, and $\sum_1^n f(x_i)[g(x_i)-g(x_{i-1})]$. When $g(x)\equiv x$, these integrals are the Riemann and the two Cauchy integrals; Gillespie (Ann. of Math. vol. 17 (1915) pp. 61–63) proved that they are equivalent. Examples show (i) that the CS integrals may exist when the RS does not; (ii) that the two CS integrals may exist and have different values; (iii) and that one CS integral may exist when the other does not. One theorem is the following: if g is non-decreasing, if f and g have no common discontinuities on the same side, and if the left (right) CS integral exists, then the RS integral exists and has the same value, the integrals being limits in the sense of increasing refinement of subdivisions. (Received November 23, 1942.)

39. Maxwell Reade: *On a theorem of Fédoroff and Binney.*

A theorem on harmonic functions, due to Fédoroff (Rec. Math. (Mat. Sbornik) N.S. vol. 40 (1933) pp. 168–179) and Binney (Trans. Amer. Math. Soc. vol. 37 (1935) pp. 254–265), has an analogue for subharmonic functions. The result is the following theorem. If $u(x, y)$ and $v(x, y)$ are continuous in the unit circle, and if $\int_R u dy - v dx \geqq 0$ and $\int_R u dx + v dy = 0$ for all oriented rectangles, R lying in the unit circle, then there exists a function $f(x, y)$ that is subharmonic in the unit circle for which $\partial f/\partial x = u(x, y)$, $\partial f/\partial y = v(x, y)$. The proof follows Levin (*Contributions to the calculus of variations*, Chicago, 1942, pp. 363–410). (Received October 17, 1942.)

40. Maxwell Reade: *Remarks on a paper of Beckenbach.*

Beckenbach's Theorem 1 (Duke Math. J. vol. 8 (1941) pp. 393–400) is similar to the classic Montel-Rado theorem characterizing functions having subharmonic logarithms; therefore it admits of Saks-Kierst types of generalizations (S. Saks, Acta Univ. Szeged vol. 5 (1930–1932) pp. 187–193). A typical result of this paper is the following theorem. Let $f(t)$ be of class c^2, with $f'(t)>0$ for $-\infty<t<\infty$. If $v(x, y)$ is of class c^2 in a domain D, and if $f(\log[(x-\alpha)^2+(y-\beta)^2]+v(x, y))$ is subharmonic for each choice of the real constants α and β, then $v(x, y)$ is subharmonic in D. In addition Beckenbach's Theorem 2 is given a slightly more general form. (Received October 16, 1942.)

41. Raphael Salem: *On some singular monotonic functions which are strictly increasing.*

The paper gives simple direct constructions of some singular monotonic functions which are strictly increasing. It contains also some new indications on Minkowski's singular function, particularly concerning its modulus of continuity. (Received October 9, 1942.)

42. H. M. Schwartz: *On sequences of Stieltjes integrals.* II.

Given a sequence $g_n(x)$ $(n=1, 2, \cdots)$ of functions of bounded variation, the

problem of determining sets $C(F)$ of conditions necessary and sufficient for the convergence of the sequence $\int_a^* f dg_n$ for all functions $f(x)$ of a given family F, is an extension of the problem treated by Lebesgue in his paper *Sur les intégrales singulières* (Ann. Fac. Sci. Univ. Toulouse vol. 23 (1909) pp. 25–117) and which corresponds to the case of absolutely continuous $g_n(x)$. Considering first the case of Riemann-Stieltjes integration over a finite interval, conditions $C(F)$ were obtained earlier (Bull. Amer. Math. Soc. abstract 48-1-55) for certain families F, where F are proper subsets of the set F^* of all functions which are defined in (a, b) and are simultaneously integrable with respect to all g_n. The principal result of the present paper is the determination of a set $C(F^*)$. In the special case of absolutely continuous g_n, this result provides a solution for the Lebesgue problem in the case, not treated by Lebesgue, when F consists of the class of Riemann integrable functions. The paper is concerned also with the extension of the results to the case of an infinite interval (a, b) and to more general Stieltjes integrals of Riemann type. (Received November 20, 1942.)

43. I. M. Sheffer: *Note on a linear transformation of "analytic" type.*

A linear problem concerning analytic functions is often expressed in terms of the power series coefficients of the functions, thus leading the problem to a system of linear equations in the coefficients: (1) $A_n[X]=\sum_{k=0}^{\infty} a_{nk}x_k=c_n$, $n=0, 1, \cdots$, where X represents the sequence $\{x_n\}$. Let $\limsup |x_n|^{1/n}$ be called the type of sequence $\{x_n\}$. As a first step in the analysis of transformation (1) examine the following question (together with some of its consequences): Under what conditions will sequences $\{x_n\}$ of given type transform into sequences of prescribed type under (1)? A typical result is the following: In order that $\{A_n[X]\}$ be of type at most h for every $\{x_n\}$ of type at most r it is necessary and sufficient that: (i) If r_n is the radius of convergence of $A_n(t)=\sum_{k=0}^{\infty} a_{nk}t^k$, and $r^*=$g.l.b. $\{r_n\}$, then $r^*>r$. (ii) Let $A_n^*(t)=\sum_{k=0}^{\infty}|a_{nk}|t^k$. For every $\epsilon>0$ there corresponds an r' in $r<r'<r^*$ such that $\{A_n^*(r')\}$ is of type at most $h+\epsilon$. (Received November 24, 1942.)

44. W. S. Snyder: *Non-parametric surfaces and inscribed polyhedra.* Preliminary report.

Let $z=f(x, y)$ define a continuous surface over an oriented rectangle of the xy-plane. Kempisty (*Sur la methode triangulaire* $\cdots$, Bull. Soc. Math. France vol. 64 (1936)) and the author have studied certain functions of rectangles associated with the surface and have derived sufficient conditions that the Burkill integrals of the functions exist and equal the Lebesgue area of the surface. This paper undertakes the study of the simple Burkill integrals (Burkill, *Functions of intervals*, Proc. London Math. Soc. (2) vol. 22 (1924)) of these functions. Necessary conditions that the simple Burkill integrals of these functions exist and equal the Lebesgue area of the surface are derived, and somewhat stronger conditions are shown to be sufficient for this result to hold. (Received November 23, 1942.)

45. W. C. Strodt: *Analytic solutions of nonlinear difference equations.* Preliminary report.

A new approach to difference equations is outlined, and applied to a large class of nonlinear difference equations. Special solutions are obtained as uniform limits of

analytic solutions of q-difference equations, equations which yield more readily to power series treatment. The given equation $P[x, y(x), y(x+\omega_1), \cdots, y(x+\omega_n)]=0$ (where the ω_k are arbitrary positive numbers, and P is a polynomial in the $y(x+\omega_k)$, analytic in x) is replaced by the approximating functional equation $P[x, y(x), y(q_1x+\omega_1), \cdots, y(q_nx+\omega_n)]=0$, where $q_i=1-\omega_i\eta$, η being any small positive number. Each transformation $x\to q_ix+\omega_i$ has $1/\eta$ for fixed point. A translation of origin to this point transforms this functional equation into the q-difference equation $P[u+1/\eta, z(u), \cdots, z(q_nu)]=0$, where $u=x-1/\eta$, $z(u)=y(u+1/\eta)$. This equation has for solution a unique power series $z(u; \eta)$ in u whose convergence is shown by the calculus of limits. The uniform limit of $z(x-1/\eta; \eta)$, as η assumes a suitable sequence of values tending to 0, is an analytic solution of the given difference equation. There is evidence that the solution generalizes to nonlinear equations Nörlund's "principal solution." The procedure applies formally to essentially every difference equation; a preliminary transformation $y(x)=x^sY(x)$ is sometimes needed. (Received October 16, 1942.)

46. Otto Szász: *On Abel and Lebesgue summability.*

The main results of this paper are: If the real numbers a_n satisfy the condition (*) $\sum_n^{2n}(|a_\nu|-a_\nu)=O(1)$ as $n\to\infty$, then Abel summability of the series $\sum_1^\infty a_n$ implies its Lebesgue summability; more explicitly the series $\sum a_n(\sin nt/n)=F(t)$ converges uniformly and $\lim_{t\downarrow 0}t^{-1}F(t)=s$ exists. At the same time the series $\sum a_n$ may diverge; in fact a series $\sum a_n$ is constructed which is Abel summable and satisfies (*), while $\limsup|a_n|=1$, so that $\sum a_n$ diverges. On the other hand it is proved that if (*) holds and if $\sum a_n$ converges, then the series $\sum a_n(\sin nt/nt)=t^{-1}F(t)$ converges uniformly in the interval $0<t<\pi$, that is, $\sum a_n$ is Lebesgue summable. It is to be noted that convergence alone does not imply Lebesgue summability. The method is similar to the one used in a previous paper: Amer. J. Math. vol. 64 (1942) pp. 575–591. (Received October 20, 1942.)

47. W. C. Taylor: *Asymptotic behavior of the Abel sums of the Laguerre expansion.*

For the Laguerre expansion $f(x)\sim\sum c_\nu L_\nu^{(\alpha)}(x)$ the behavior of the Abel sum $A(x,w)=\sum c_\nu L_\nu^{(\alpha)}(x)w^\nu$ is studied. Conditions are obtained which insure that $A(x, w)-f(x)$ is $o(1)$ or $o(f(x))$ under various limiting processes involving $x\to\infty$ and/or $w\to 1$. (Received November 21, 1942.)

48. W. J. Trjitzinsky: *Singular nonlinear integral equations.*

The author studies problems (1) $\phi(x)+\int K(x, t)f(t, \phi(t))dt=0$, (2) $\phi(x)+\int\Gamma(x, y, \phi(y))dy=0$ (ϕ is the unknown; the integrals are definite), where $\Gamma(x, y, z)=\int K(x, t)k(t, y, z)dt$. The functions $f(t, u)$, $k(t, y, u)$ are subject to Lipschitz conditions in u. Past investigations were under the supposition that the kernel $K(x, t)$ is regular in the sense that the characteristic values of $K(x, t)$ form a discrete set, Fredholm theory being applicable. The author proceeds with more general kernels, not regular in the above sense, belonging to L_2 in each of the variables separately; such kernels are singular. Among other results a number of existence theorems for (1), (2) are established. These problems are of importance in themselves as well as in many phases of mathematical physics; also, many nonlinear differential problems with linear boundary conditions are reducible to the problem (1). This theory naturally depends in an

essential manner on the theory of singular kernels as previously developed by Carleman and later by the present author. (Received November 20, 1942.)

49. S. M. Ulam: *On equivalence of functions.*

Two functions, $f(x)$ and $g(x)$ defined on $0 \leqq x \leqq 1$ are said to be equivalent, if there exists a one-to-one transformation of the interval $h(x)$ such that $f(x) = hgh^{-1}(x)$. This notion of equivalence is investigated under restriction of $f(x)$ and $g(x)$ to various classes of functions. Among others, it is proved that if $f(x)$ is a one-to-one analytic function, it is equivalent to a Baire function of finite class. Necessary and sufficient conditions are given for equivalence of two continuous functions—and an enumeration of possible equivalence types of such functions is obtained. (Received November 23, 1942.)

50. Wolfgang Wasow: *On a boundary layer problem for a certain linear partial differential equation.*

Let $u(x, y, \lambda)$ be the function satisfying the differential equation $(1/\lambda)\Delta u + u_x = f(x, y)$ in the interior of a convex domain B and assuming prescribed boundary values, not depending on λ, on the boundary C of B. As $\lambda \to \infty$ $(\lambda > 0)$ the differential equation reduces to the limiting differential equation $u_x = f(x, y)$. In the paper the author proves that $v(x, y) = \lim_{\lambda \to \infty} u(x, y, \lambda)$ exists and satisfies the limiting differential equation. But in general $v(x, y)$ will satisfy the prescribed boundary condition only on a certain arc C_1 of C. Along the remaining arc C_2 of C the convergence of $u(x, y, \lambda)$ is non-uniform and gives rise to a boundary layer phenomenon. The proof is based on certain estimates for Green's function and uses an idea suggested in a paper by E. Rothe. (See E. Rothe, *Asymptotic solution of a boundary value problem*, Iowa State College Journal for Science vol. 13 (1939).) (Received November 23, 1942.)

51. Alexander Weinstein: *On a general variational method for the determination of eigenvalues.*

Let $U(v)$ and $H(v)$ be two positive definite quadratic functionals and let P be the eigenvalue problem relative to the variational problem $U(v)/H(v) = \text{Min.}$ under certain boundary conditions BC which express the vanishing of v, dv/dn, and so on, on certain parts of the boundary. Let P_0 be the eigenvalue problem obtained from P by cancelling a certain number of the boundary conditions BC. The eigenvalues in P_0 are obviously not greater than those in P. It can be shown that the eigenvalues in P can be expressed in terms of the solutions of P_0. This result is obtained by linking P_0 with P by a chain of intermediate problems $P_1, P_2, \cdots$ defined in a similar way as those used in particular cases by the author (Mémorial des Sciences Mathématiques, no. 88). (Received November 16, 1942.)

52. J. E. Wilkins: *A class of functions in the calculus of variations for multiple integrals in parametric form.*

In order that a multiple integral with integrand $f(y^1, \cdots, y^n, p_1^1, \cdots, p_m^n)$, where $p_a^i = \partial y^i / \partial t_a$, be independent of the parametric representation $y^i = y^i(t)$ of the variety for which it is computed, it is necessary and sufficient that the equation $f(y, pA) = |A| f(y, p)$ hold for every m-rowed square matrix A with positive determi-

nant. It was proved in the author's doctoral dissertation that there exist constants $N_{\alpha;\gamma}^{\beta;\epsilon}$ such that the equation, $p_\omega^\iota f_{\iota\kappa}^{\beta\alpha} = M_{\alpha;\gamma}^{\beta;\epsilon} f_\epsilon^\gamma$ holds for every function f satisfying the above relation. In this equation $\iota=(i_1, \cdots, i_g)$, $\alpha=(a_1, \cdots, a_g)$, $\beta=(b_1, \cdots, b_g)$, $\epsilon=(e_1, \cdots, e_h)$, $\kappa=(k_1, \cdots, k_h)$, $\gamma=(c_1, \cdots, c_h)$, p_α^ι stands for the product of the corresponding p's, and f_ϵ^γ stands for the derivative of f with respect to the corresponding p's. It is the purpose of this paper to give an explicit definition of the constants $M_{\alpha;\gamma}^{\beta;\epsilon}$ in terms of a certain set of permutations. On the basis of this definition certain properties of the M's are readily deduced. (Received October 27, 1942.)

53. J. E. Wilkins: *A note on the Weierstrass condition for multiple integrals in the calculus of variations.*

For the purpose of obtaining a sufficiency theorem for multiple integral problems in the calculus of variations, Carathéodory has introduced an E-function $E_c(x, y, p, P)$ different from the usual E-function E_0. It is proved here that $E_c=E_0$ in case $P-p$ has rank one. Consequently, a necessary condition for a minimum is that $E_c \geqq 0$ for all P such that $P-p$ has rank one. But an example is given to show that it is not true that $E_c \geqq 0$ for all P is necessary for a minimum. (Received October 30, 1942.)

54. J. E. Wilkins: *Definitely self-conjugate adjoint integral equations.*

This paper contains a new definition for definitely self-conjugate adjoint integral equations which is substantially weaker than that originally given by Reid. For such systems it is found that all of the expansion theorems obtained by Reid remain valid, but that there need not now be a denumerable infinity of characteristic values. Moreover it is shown that the index of a characteristic value is equal to its multiplicity as a zero of the Fredholm determinant. (Received November 25, 1946.)

55. Fumio Yagi: *On a certain Stieltjes integral equation.*

Let $g(z)$ and $p(\lambda)$ be two given functions which satisfy certain norm conditions. It is known (Cameron-Martin, *Infinite linear difference equations with arbitrary real spans and first degree coefficients*, to be published) that the integral equation $\int_{-\infty}^{\infty} f(z-\lambda)dp(\lambda)=g(z)$ has a unique analytic solution in $a<\mathrm{Im}\, z<b$ of the same norm as $g(z)$, provided that $P(w)=\int_{-\infty}^{\infty} e^{w\lambda}dp(\lambda)$ is non-vanishing in a strip $c<\mathrm{Re}\, w<d$. In this paper the author allows the function $P(w)$ to have a finite number of zeros in $c<\mathrm{Re}\, w<d$ and shows that there exists an analytic solution $f(z)$ in $a<\mathrm{Im}\, z<b$ of the same norm as $g(z)$. Moreover it is proved that every other solution of the required type is the form $f(z)+\sum_{n=1}^{N}\sum_{m=1}^{a_n} A_{nm} e^{-w_n z} z^{m-1}$ where A_{nm} are constants, w_n $(n=1, 2, \cdots, N)$ are the zeros of $P(w)$, and a_n is the order of zero at w_n. (Received November 20, 1942.)

56. J. W. T. Youngs: *On the additivity of the Lebesgue area.*

Suppose a surface S is given in parametric form by the continuous triple $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ with (u, v) in the unit square Q. Each division of Q into two rectangles R_1 and R_2 by a line yields a decomposition of S into two surfaces S_1 and S_2. This paper provides a direct proof for the theorem that, if the continuous triple carries the dividing line into a single point, then the Lebesgue area is additive:

$L(S)=L(S_1)+L(S_2)$ (Rado and Reichelderfer, *On a stretching process for surfaces*, Amer. J. Math. vol. 61 (1939)). (Received November 28, 1942.)

57. Antoni Zygmund: *A property of the zeros of Legendre polynomials.*

Suppose that $n<m$ are positive integers and that a polynomial $\phi(x)$ of degree n does not exceed M in absolute value at the zeros of the Legendre polynomial $P_m(x)$. Then $|\phi(x)| \leq A(\delta)M$ for $-1 \leq x \leq +1$, where $A(\delta)$ depends only on the number δ defined by the equations $m/n=1+\delta$. Similar results hold for the integrals $\int_{-1}^{+1}|\phi(x)|^r dx$ with $r \geq 1$. (Received November 23, 1942.)

Applied Mathematics

58. Stefan Bergman: *A formula for the stream function of compressible fluid flow.*

Let $\mathfrak{q}=ve^{i\theta}$ denote the velocity vector. A flow, $\mathcal{J}$ is said to be of the type D_n, if the boundary of the domain, B, in which $\mathcal{J}$ is defined consists of $2n$ segments S_K such that along each S_{2K}, $K=1, 2, \cdots, n$, $\theta=\theta_K$ is constant and along each S_{2K-1}, v is constant. (S_{2K} are segments of straight lines, S_{2K-1} are so-called "free boundaries.") The image of B in the logarithmic plane (see *Notes on hodograph method in the theory of compressible fluid*, publication of Brown University, p. 6) is a polygonal domain. In the case of an incompressible fluid the stream function of $\mathcal{J}$ can be represented as a closed expression with n parameters. The author considers subsonic flows, $\mathcal{C}$, of compressible fluid. Using certain linear operators (see above mentioned *Notes*, §§6 and 10, and Trans. Amer. Math. Soc. vol. 53 (1943) pp. 130–155) he derives a similar explicit formula with n parameters for the flows $\mathcal{C}$ of "nearly type D_n," that is to say, for flows whose boundaries consist of $2n$ segments along which θ or v assume nearly constant values. The angles θ_K may be prescribed. (Received November 21, 1942.)

59. R. M. Foster: *On the average resistance of an electrical network.*

In an electrical network composed of two-terminal resistance elements, let J_i designate the total resistance measured across the terminals of an element, the internal resistance of this element being r_i, and let S_i be the driving-point resistance measured in the branch containing this element r_i. It is shown in this paper that, for any network configuration whatsoever, $\sum J_i/r_i=R$ and $\sum r_i S_i=N$ (the summation being extended over all the elements of the network), with $R=V-P$ and $N=E-V+P$, where E is the number of elements, V the number of vertices, and P the number of separate, unconnected parts of the configuration. The average values of the ratios J_i/r_i and r_i/S_i are thus R/E and N/E, respectively. If all the elements of the network have the same internal resistance r, and if there is complete symmetry among the elements so that the resistance measured across any one element is necessarily equal to that across any other element, then $J_i=rR/E$ and $S_i=rE/N$. These results are extended to generalized impedances, and to infinite networks. (Received November 23, 1942.)

60. A. H. Fox: *Integral representation of the flow of a compressible fluid around a cylinder.*

The steady irrotational two-dimensional flow of a compressible fluid may be approximated by the flow of a hypothetical incompressible fluid in which the pressure is

represented by a polynomial in the reciprocal of the density. The stream function of the flow may be expanded in a series in terms of a parameter q that appears in the polynomial. For $q=0$ this reduces to the von Kármán-Tsien method of attacking similar problems. The differential equation of the flow in the hodograph plane may be reduced to the form $\psi^*_{z\bar{z}}+F\psi^*=0$, for which the integral representation ψ^* is the real part of $\int_{-1}^{1}E^*(z, \bar{z}, t)(1-t^2)^{-1/2}f[1/2z(1-t^2)]dt$ has been established by Bergman. The function f is a function of one complex variable determined by the pattern of the flow in the hypothetical fluid. In the present paper, the method is applied to the flow of compressible fluid around a circular obstacle. (Received November 24, 1942.)

61. K. O. Friedrichs: *On nonlinear vibrations of third order.*

Problems of self-excited oscillations frequently lead to differential equations of higher than second order. While for the order two, Bendixson's theorem furnishes a general criterion for the existence of periodic solutions, no similar general criterion is available in cases of higher order. The present paper analyzes such a problem of order three (occurring in the theory of electrical oscillations in circuits involving vacuum tubes). First, the existence of periodic solutions is obtained from the fixed-point theorem by a special construction. In addition, periodic solutions are investigated in the neighborhood of degenerate cases of order two by a perturbation method, which yields asymptotic expansions. (Received November 23, 1942.)

62. Hilda Geiringer: *New convergence cases for iteration methods applied to linear equations.*

To obtain the solution $x^{\rightarrow}$ of the system $\sum_{k=1}^{n}a_{ik}x_k+r_i=0$ $(i=1, \cdots, n; |A| \neq 0)$, by iteration through "successive displacements" new values $x_i^{(\nu+1)}$ are successively computed from the ith equation using $x_1^{(\nu+1)}, \cdots, x_{i-1}^{(\nu+1)}, x_{i+1}^{(\nu)}, \cdots, x_n^{(\nu)}$, $(\nu=0, 1, 2, \cdots; x^{(0)\rightarrow}$ arbitrary). (See Bull. Amer. Math. Soc. abstract 48-5-202.) The only sufficient condition for the convergence of $x^{(\nu)\rightarrow}\rightarrow x^{\rightarrow}$ known so far is that the matrix (a_{ik}) is positive definite. In this case the procedure reduces to Southwell's relaxation-method. Now the necessary and sufficient condition for convergence is that the roots ρ_i of a certain equation $F_{n-1}(\rho)=0$ of degree $n-1$ are all less than one in absolute value, a condition assuming no symmetry. If, however, the matrix is symmetric and $a_{ii}>0$ then positive-definiteness is necessary for convergence from an arbitrary starting point $x^{(0)\rightarrow}$. In addition it is proved that all sufficient conditions known for the "ordinary" iteration, that is, for the iterated linear homogeneous transformation of the "error-vector" $z_i^{(\nu)}=x_i^{(\nu)}-x_i$, assure convergence of the successive displacements, but by no means vice versa; and a new sufficient condition for both methods is established. In all these cases any order is admissible for the successive displacements; also the convergence may be improved by "group-iteration." (Received November 23, 1942.)

63. A. E. Heins: *Some remarks on the solution of dual integral equations.* I.

The solutions of dual integral equations whose kernels can be expressed in a series of Bessel functions are considered. Explicit solutions for these equations are given and their properties are discussed. (Received November 23, 1942.)

64. Max Herzberger: *Direct methods in geometrical optics.*

The author presents a new approach to the problem of geometrical optics deviating

from the methods of W. I. Hamilton. Hamilton has shown the existence of a characteristic function, the knowledge of which gives all the information necessary about an optical system. However, the construction of Hamilton's function for any given optical system seems to lead to insurmountable mathematical difficulties. The author proposes instead to calculate the coordinates of the image rays directly as functions of the coordinates of the object rays. Instead of getting a single function like Hamilton, he finds four functions connected by three differential equations and one finite equation. He has succeeded in obtaining the explicit form of these functions for any given optical system and developing an image error theory which is more adapted to the practical problems of lens design. (Received October 3, 1942.)

65. Fritz John: *Instability of certain nonlinear vibrations.*

The differential equation (E): $x''(t)+f(x)=-F\sin\lambda t$, where $f(0)=0$, g.l.b. $f'(x)>0$, λ and F positive constants, represents the motion of a spring with nonlinear restoring force and without damping under a periodic disturbing force. It is shown here: For every $M>0$ there exists and is uniquely determined a value $\lambda>0$ and a solution $x(t)$ of (E) of period $2\pi/\lambda$, for which $x=M$, $x'=0$ for $t=\pi/2\lambda$ and $x'>0$ for $|t|<\pi/2\lambda$ ($x(t)$ then is a periodic solution of phase difference π/λ). The value λ determined by M may define the "resonance function" $\lambda=\lambda(M)$ (this is only one branch of the complete resonance curve). For $d\lambda/dM>0$ the corresponding periodic solutions $x(t)$ are shown to be unstable in the following sense: there is an $\epsilon>0$, such that for every $\delta>0$ there exists a solution $x_1(t)$ of (E), for which $|x_1-x|<\delta$, $|x_1'-x'|<\delta$ for some t, and $|x_1-x|>\epsilon$ for other t. One always has $d\lambda/dM\leqq 0$ for "soft" springs, that is, if $f'(x)\leqq f(x)/x$. (Received November 23, 1942.)

66. Edward Kasner: *Trajectories in a resisting medium.*

Consider the motion of a particle moving in a general positional field and influenced by a resisting medium R. The five fundamental properties of a pure positional field (*Differential-geometric aspects of dynamics*, Amer. Math. Soc. Colloquium Publications vol. 3, 1913, chap. 1) are of course in general changed. However if $R=Av^2+B$, property I remains valid. If properties I and II are valid then the resistance is found to vary as the square of the speed. If I, II, III hold then R vanishes. The discussion is carried out for both two- and three-dimensional fields. Fields of simple and complicated character are studied where either all, or the maximum infinitude, of trajectories are circles (a particular example is the Maxwell central force). (Received November 18, 1942.)

67. Arthur Korn: *On vibrational vortices.*

The vortex equations in fluids are nonlinear differential equations, but in some special cases they may be replaced by linear equations. A special case is the problem of vibrational velocities (vectors): $u=u_0+u_1\cos\nu t+u_2\sin\nu t$, where ν is an exceedingly great frequency and the first derivatives of u_0, u_1, u_2 with respect to the time t and the first derivatives with respect to the coordinates x, y, z of u_1, u_2 can be neglected in comparison with νu_0, νu_1, νu_2. Then one remarks that on the left-hand side of the vortex equations the main terms are $(-\text{curl } u_1\sin\nu t+\text{curl } u_2\cos\nu t)\nu/\rho$, ρ being the density of the fluid. In order to obtain on the right-hand side such great terms, the first derivatives of u_0 with respect to x, y, z must be exceedingly great. Excluding exceedingly great values of $\rho u_0^2/2$ such cases cannot be realized in incompressible fluids, but they are perfectly possible in compressible fluids, even if the conditions

div $u_1=0$ and div $u_2=0$ are retained. u_0 must have wave character with an exceedingly small wave length λ. Such cases can be handled by means of linear differential equations. (Received November 21, 1942.)

68. Brockway McMillan: *Networks of mechanisms.* Preliminary report.

A mechanism M maps a class I of input histories $i(t)$ upon a class O of output histories $o(t)$, $-\infty<t<\infty$. It is single-valued and has the property that whenever $i_1(t)\equiv i_2(t)$ for $t\leqq t_0$, then $o_1(t)\equiv o_2(t)$ for $t\leqq t_0$. If $o_1(t)\equiv o_2(t)$ for $t\leqq t_0+\lambda$, uniformly in i_1, i_2, and t_0, then M has the latency λ. Suppose that I is closed under an operation of addition, that a null function $i_0(t)\equiv\phi$ is in I, that every $i(t)$ is identically ϕ near $t=-\infty$, and that $O\subseteq I$. Let (M_k) be a collection of mechanisms from I to O such that (a) each has latency at least $\lambda>0$, and (b) each maps the null function on itself. The inputs and outputs of the M_k are interconnected to form a network N. Arbitrary inputs $i(t)\in I$ at each junction make the components of a vector input to N. The vector output of N has for its components the outputs of the various M_k. Theorem: I and O can be extended so that N is a mechanism between its vector inputs and outputs, with latency λ. A motivation is the possibility of application to nerve fiber networks. (Received November 4, 1942.)

69. W. H. Roever: *A new formula for the deviation in range of a projectile due to the earth's rotation.*

On page 68 in his monograph entitled, *The weight field of force of the earth*, published in the Washington University Studies, September, 1940, the author derives for the range of a projectile, a formula [second part of (129)] which by a simple trigonometric transformation can be put in the new form $\bar{x}=(v_0^2/g_1)\sin 2\beta+\Delta\bar{x}$ where $\Delta\bar{x}=-(4v_0^3/3g_1^2)\,\omega\cos\phi_1\sin 3\beta\sin\alpha$, in which ω is the angular velocity of the earth's rotation, g_1 is the acceleration, due to weight, at the position of the gun, ϕ_2 is the astronomical latitude of the position of the gun, α is the azimuth (measured from the south through the west) of the direction of fire, β is the angle of elevation of the gun, v_0 is the muzzle velocity of the projectile, he points out particularly that for fixed values of α and ϕ_1, $\Delta\bar{x}$ changes sign when $\beta=60°$. (Received November 23, 1942.)

GEOMETRY

70. John DeCicco: *Conformal geometry of second order differential equations.*

Kasner in his fundamental paper, *Conformal geometry*, Prcoeedings of the International Congress of Mathematicians, 1912, initiated the conformal study of sets of analytic curves. In previous work, Kasner (with the author) studied the conformal geometry of velocity systems of curves $y''=(1+y'^2)[\phi(x,y)+y'\psi(x,y)]$. This class of velocity systems characterizes the conformal group. Any velocity system possesses six absolute conformal differential covariants of second order. In this paper, it is shown that a system of ∞^2 curves, not of the velocity type, possesses three absolute conformal differential covariants of third order. Moreover any other conformal covariant is a function of these and their partial derivatives. Geometric interpretations of these covariants are also obtained. (Received November 21, 1942.)

71. Jesse Douglas: *Point transformations and isothermal families of curves.*

A classification is made of all point transformations $U=f(u, v)$, $V=g(u, v)$ ($u=x+iy$, $v=x-iy$) with respect to the isothermal families of curves ($\phi(u)+\psi(v)=$const.) which they convert again into isothermal families. An incidental result is a very simple proof of the theorem that only conformal transformations, $U=f(u)$, $V=g(v)$, convert all isothermal families into isothermal families. (A proof of a more general theorem, involving arbitrary lineal element transformations, has been given recently by Kasner.) Canonical transformations found are (1) $U=u$, $V=u+v$, which preserves the isothermal character of $V+\phi(U)=$const. (ϕ arbitrary); (2) $U=u+v$, $V=u-v$, preserving the isothermality of all curve families equivalent by translation and similitude to that defined by sn $U\,dU+$sn $V\,dV=0$. Here sn is Jacobi's elliptic function with any modulus k. Interesting special cases (trigonometric, algebraic) are obtained for particular values of k. The given canonical transformations may be pre- and post-multiplied by arbitrary conformal transformations. The problem may be rendered topological by referring it to the determination of the common diagonal curves of two given nets. (Received November 23, 1942.)

72. Edward Kasner: *The close packing of spheres.*

If space is packed with equal spheres the volume occupied is about seventy-five per cent and the porosity about twenty-five per cent. By using many spheres of unequal radii, it is proved that the porosity can be made as near zero as desired. An analogous theorem holds for circles in a plane, or for hyperspheres in any space. (Received October 9, 1942.)

73. Edward Kasner and John DeCicco: *Bi-isothermal systems in pseudo-conformal geometry.*

In this paper, generalizations are obtained of Kasner's theorems on biharmonic functions in the paper, *Biharmonic functions and certain generalizations*, Amer. J. Math. vol. 58 (1936) pp. 377–390. A bi-isothermal system of hypersurfaces in euclidean four-dimensional space is pseudo-conformally equivalent to ∞^1 parallel hyperplanes. Similarly a bi-isothermal system of curves is pseudo-conformally equivalent to ∞^3 parallel lines. Any bi-isothermal system of hypersurfaces is intersected by a conformal surface in an isothermal system of curves. If every bi-isothermal system of hypersurfaces intersects a given surface s is an isothermal system, then s is conformal. If a system S of ∞^1 hypersurfaces is intersected by every conformal surface in an isothermal system, then S is bi-isothermal. Kasner's pseudo-angle between any two bi-isothermal systems of hypersurfaces and curves is a biharmonic function. The complete group of point transformations preserving the class of biharmonic systems of hypersurfaces (or curves) is the mixed pseudo-conformal group. (Received November 21, 1942.)

74. Mary E. Ladue: *Conformal geometry of horn angles of higher order.*

The conformal measures of horn angles (curvilinear angles of zero magnitude) between curves having third, fourth and fifth order contact are obtained. The conformal geometry of the third order horn set (the set of all curves having third order contact with each other at a given point) is studied by making correspond to each

curve of the horn set a point in a four-dimensional space K_4 in which the metric is given by the conformal measure of a horn angle of the horn set, and the transformation is that induced on the points of the K_4-space by a conformal transformation on the curves of the horn set. The invariants of the geometry of this K_4-space are then obtained. Finally the equilong measure (see Kasner, *Equilong invariants and convergence proofs*, Bull. Amer. Math. Soc. vol. 23 (1917) pp. 341–347) of horn angles of third order contact is obtained and the resulting equilong geometry of the horn set is shown to be isomorphic to the conformal geometry of the corresponding horn set. Similar results for the equilong geometries of horn sets of fourth and fifth order are to be presented soon. (Received November 20, 1942.)

75. Karl Menger: *Statistical generalizations of metric geometry.*

A probability function $\Pi(x; p, q)$ instead of a number is associated with each ordered pair of elements p, q of a set S and interpreted as the probability that the point p has a distance at most x from the point q. Assume as postulates $\Pi(0; p, q)=1$ or $\neq 1$; according to whether $p=q$ or $p\neq q$; $\Pi(x; p, q)=\Pi(x; q, p)$; and a triangular inequality $\Gamma(\alpha, \beta)\leqq\gamma$ connecting $\alpha=\Pi(x; p, q)$, $\beta=\Pi(y; q, r)$, and $\gamma=\Pi(x+y; p, r)$ where $0\leqq\Gamma(\alpha, \beta)=\Gamma(\beta, \alpha)\leqq 1$ and Γ is a non-decreasing function such that $\Gamma(1, 1)=1$ and $\Gamma(\alpha, 1)>0$ if $\alpha>0$. Define q to be between p and r and write pqr if $\Gamma(1-\alpha, 1-\beta)\leqq 1-\gamma$. Then it can be shown that (1) pqr implies rqp; (2) if q is distant from r, that is, if for some $y>0$, $\Pi(y; q, r)=0$, then pqr and prq are incompatible; (3) under certain conditions, pqr and prs imply pqs and qrs. If S contains more than four elements it can be ordered by means of the above defined betweenness relation. (Received October 28, 1942.)

76. P. M. Pepper: *A new method for imbedding theorems.*

Let Σ be a system of sets with a congruence relation for pairs of elements of Σ, and let E be an element of Σ such that each $n+4$ or more point element of Σ which is not congruent to a subset of E has at least $m+1$ of its $n+3$ point subsets not congruent to subsets of E. Then for $k\geqq 1$, each $n+3+k$ or more point element of Σ which is not congruent to a subset of E has (in terms of the binomial coefficients $C_{n,r}$) at least $1+mk+(m-1)C_{k,2}+\cdots+C_{k,m}$ of its $n+3$ point subsets not congruent to subsets of E. If Σ is the class of all semimetric spaces, the above theorem, in conjunction with a finite covering theorem, permits a new proof of a sharp imbedding theorem for the euclidean n-space under weakened hypotheses and a comparable new result for a convex n-sphere. (Received October 29, 1942.)

77. E. J. Purcell: *Flat space congruences of order one in* $[n]$.

By an $[n-k]$-congruence of order one in $[n]$, $(k<n)$, is meant an algebraic ∞^k-system of $[n-k]$'s in n-dimensional projective space such that one and only one $[n-k]$ of the system passes through an arbitrary point of $[n]$. This paper defines, classifies, and studies 2^{k-1} types of $[n-k]$-congruences of order one in $[n]$, for which the fundamental loci on a generic $[n-k]$ are all distinct. (Received November 2, 1942.)

78. W. H. Roever: *The axonometric method of representing the points of space on a plane.*

After stating that axonometry is one of the modern methods of descriptive

geometry which is but little known in this country, the author describes his method of representing space upon the plane as follows:—Adjoin to the object to be represented its shadow on some fixed plane, such as the ground, and then project this augmented figure on the picture plane, thus obtaining thereon for each point P of the object its projection P^* and also the projection P'^* of its shadow. He thus obtains for a point P of space the two related picture-plane points P^* and P'^* which together furnish enough information to obtain therefrom definitely and uniquely the position of P in space. He also shows that this method yields the same result as that given by him in his monograph entitled *Fundamental theorems of orthographic axonometry and their value in picturization* (Washington University Studies, n.s., Science and Technology, no. 12, St. Louis, 1941). (Received November 19, 1942.)

79. Domina E. Spencer: *The tensor representation of the figures of Study's "Geometrie der Dynamen."*

In an earlier paper (*Geometric figures in affine space*, Bull. Amer. Math. Soc. abstract 48-9-281) the affine ancestors of the Study figures were investigated and the foundations were laid for the tensor representation of the figures themselves. The present paper continues the work by the introduction of a metric and the detailed study of the various Study figures. This systematic approach to the subject will make possible important applications which were obscured by the complexity of the original treatment of Study. (Received November 18, 1942.)

80. C. E. Springer: *Dual geodesics on a surface.*

In this paper a dual geodesic is defined to be a curve on a metric surface with the property that its ray-point corresponding to every point P on the curve lies on the line l_2 which is in Green's Relation R to the normal line l_1 to the surface at P. The differential equation of the dual geodesics is derived. The directions of Segrè are characterized as the directions in which the geodesics and dual geodesics coincide. Finally, the cubic curve, which is the locus of the ray-points of geodesics through a point of the surface, is studied. (Received October 27, 1942.)

81. S. M. Ulam: *On the length of curves, the surface area and the isoperimetric problem under a general Minkowski metric.* Preliminary report.

Given a metric in Euclidean space, defined through a symmetric convex unit gauge, one is led to the notion of length of curves through the usual process of polygonal approximation. The isoperimetric problem in the plane has a solution. A notion of surface area, invariant with respect to congruence under the given metric, is introduced. (Received November 23, 1942.)

82. J. E. Wilkins: *The first canonical pencil.*

It is the purpose of this paper to give several geometric definitions for a general canonical line of the first kind. Each such canonical line except the first axis of Čech may be defined in terms of the cusp-axes of the two families of hypergeodesics which are extremals of the integrals $\int \beta^n \gamma^{1-n} v'^{2-2n} du$, $\int \gamma^n \beta^{1-n} v'^{2n-1} du$, where n is constant, or as the cusp-axis of a cone of class three which is defined by means of the osculating planes of these hypergeodesics. If at each point of the surface is introduced the triple of directions $\overline{D}_k$ conjugate to the directions D_k considered by Bell (Duke Math. J.

vol. 5 (1939) pp. 784–788), the general canonical line as the line of intersection of the osculating planes of the projective geodesics in the directions $\overline{D}_k$ results. (Received October 2, 1942.)

83. Y. C. Wong: *Some Einstein spaces with conformally separable fundamental tensors.*

When the fundamental tensor ${}^*g_{\alpha\beta}$ of a Riemannian m-space is of the form ${}^*g_{ij}=[\rho(x^a)]^{-2}g_{ij}(x^k)$, ${}^*g_{ip}=0$, ${}^*g_{pq}=[\sigma(x^a)]^{-2}g_{pq}(x^r)$, $\alpha,\beta=1,\cdots,m$; $i,j,k=1,\cdots,n$; $p,q,r=n+1,\cdots,m$, it is said to be conformally separable; ${}^*g_{ij}$ and ${}^*g_{pq}$, with x^r and x^k, respectively, as parameters, are called its component tensors. The author studies in this paper the conformally separable tensor which is the fundamental tensor of an Einstein m-space and each of whose component tensors either is of dimension less than three or is a family of fundamental tensors of Einstein spaces. It is found that the constructions of such a conformally separable tensor is invariably reduced to the construction of the fundamental tensor g_{ij} of an Einstein n-space or a Riemannian 2-space for which the equation $y_{,ij}=-Ig_{ij}$ admits a non-constant solution for y, where the comma denotes covariant differentiation with respect to g_{ij} and I is an unspecified scalar. The author is content with this result, because the latter problem has already been considered in detail by H. W. Brinkmann in his study of Einstein spaces which are conformal to each other. (This paper will be published in the Trans. Amer. Math. Soc.) (Received October 2, 1942.)

Numerical Computation

84. H. E. Salzer and Abraham Hillman: *Exact values of the first 120 factorials.*

Due to their fundamental importance, the exact values of the first 120 factorials were computed and checked. 120! contains 199 digits. 100! agreed with Uhler's value (Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) p. 61). When these values were compared with Potin's table of the first 50 factorials (*Formules et tables numériques*, p. 836) errors were found in Potin's values for 18!, 38!, 45!, and 50!. (Received November 11, 1942.)

Statistics and Probability

85. J. H. Curtiss: *A note on the theory of moment generating functions.*

The moment generating function (m.g.f.) of a variate X is defined as the mean value of $\exp(\alpha X)$, the characteristic function (c.f.) as the mean value of $\exp(itX)$, where α and t are real. The purpose of this note is to place on record careful statements and proofs of the appropriate analogues for the m.g.f. of the well known uniqueness and limit theorems for the c.f. For example, Levy's continuity theorem assumes the following form: Let $F_n(x)$ and $G_n(\alpha)$ be, respectively, the d.f. and m.g.f. of a variate X_n. If $G_n(\alpha)$ exists for $|\alpha|<\alpha_1$ and for all $n\geqq n_0$, and if there exists a function $G(\alpha)$ defined for $|\alpha|\leqq\alpha_2<\alpha_1$, $\alpha_2>0$, such that $\lim_{n\to\infty}G_n(\alpha)=G(\alpha)$ uniformly, $|\alpha|\leqq\alpha_2$, then there exists a variate X with d.f. $F(x)$ such that $\lim_{n\to\infty}F_n(x)=F(x)$ uniformly in each finite interval of continuity of $F(x)$. The m.g.f. of X exists for $|\alpha|\leqq\alpha_2$ and is equal to $G(\alpha)$ in that interval. (Received October 9, 1942.)

86. W. K. Feller: *On a probability limit theorem of H. Cramér.*

Let $\{X_k\}$ be mutually independent random variables whose first moments vanish and whose second moments are σ_k^2; let $s_n^2=\sigma_1^2+\cdots+\sigma_n^2$. In various applications one is concerned with the distribution function $Pr\{X_1+\cdots+X_n>xs_n\}$, where $x\to\infty$ as $n\to\infty$. The simplest binomial case has been studied by A. Khintchine, P. Lévy, Smirnoff and others. H. Cramér found a complete description of the asymptotic behavior of the above sums in the case where all X_k have the same distribution function. This theorem is generalized to the case of unequal components. The theorem is to serve as a base for a solution of the "problem of the iterated logarithm" in the general case. (Received November 21, 1942.)

87. W. K. Feller: *On the general form of the so-called law of the iterated logarithm.*

Let $\{X_k\}$ be a sequence of mutually independent random variables whose first moments vanish and whose second moments are σ_k^2; let $s_n^2=\sigma_1^2+\cdots+\sigma_n^2$. A sequence of numbers $\phi_k\uparrow\infty$ is said to be of upper (lower) class if the probability that $X_1+\cdots+X_n>s_{n_k}\phi_{n_k}$ for infinitely many k is one (zero); any sequence $\{\phi_k\}$ is either of upper or of lower class. A n.a.s. condition is found for a sequence $\{\phi_k\}$ to belong to the upper class. It generalizes the condition found by Kolmogoroff and Erdös in the special case where X_k assumes the values ±1 only, each with probability 1/2; however, it is different in form. The new theorem also contains a result of Marcinkiewicz and Zygmund on the necessity of the condition imposed by Kolmogoroff on the X_k in his proof of the law of the iterated logarithm. (Received November 21, 1942.)

88. P. G. Hoel: *On indices of dispersion.*

The sampling distribution of the index of dispersion for binomial and Poisson distributions is investigated by means of semi-invariants. Approximations to terms of order N^{-2} are obtained for the descriptive moments of the distribution, by means of which the accuracy of the chi-square approximation can be determined. (Received October 30, 1942.)

TOPOLOGY

89. R. F. Arens: *Homeomorphism groups of a space.* Preliminary report.

Let A be a locally bicompact, locally connected Hausdorff space, and let G be a group of homeomorphisms of A. Then there is a certain topology N making G into a topological group (Pontrjagin, *Topological groups*, Princeton, 1939). This topology N is the weakest admissible topology that can be introduced into G, in this sense: Sets in G open by N are open by any other admissible topology M. A topology M for a group of homeomorphisms, G, of a space A is called admissible if by using that topology for G the two functions $g(a)$ and $g^{-1}(a)$, where $g\in G$ and $a\in A$, become continuous functions of both arguments g and a simultaneously. The topology N is determined by the following system of neighborhoods of the identity in G: Select in A an open set W whose closure is bicompact, and another open set whose closure K is contained in W. Then the set U of all $g\in G$ which transform K into W is defined to be a neighborhood of the identity. The set of all such U together with all their finite

intersections form a complete system of neighborhoods for the identity homeomorphism in G. (Received November 16, 1942.)

90. G. D. Birkhoff: *Measure-preserving transformations of a planar ring without planar periodic points.*

Let there be given a planar ring $R: a \leqq r \leqq b$ and a one-to-one continuous measure-preserving (that is, conservative) direct transformation T of R into itself which has the same rotation number α along the circular boundaries C_a and C_b, so that it is not possible to infer the existence of periodic point groups by the aid of Poincaré's last geometric theorem. The note discusses the conclusions which may be drawn if in fact there are no periodic points whatsoever, in the special case $T = RU$ ($R^2 = U^2 =$ identity); this special hypothesis as to the form of T is satisfied in the transformation T associated with the restricted problem of three bodies. Among other things, it is proved that there will then exist a qualitative integral $\Theta(P) =$ constant such that $\Theta[T(P)] = \Theta(P) + \alpha$ where Θ is continuous in the polar coordinates (r, θ) of P and increases by 2π when P makes a positive circuit of the ring R. (Received December 1, 1942.)

91. D. G. Bourgin: *On a theorem of Goldstine's.*

A new, topological proof is given for Goldstine's theorem (Duke Math. J. vol. 4 (1938), pp. 125–131) relating $w(E)$ completeness and reflexiveness in a Banach space. (Received November 24, 1942.)

92. D. G. Bourgin: *Quasi norms in linear topological spaces.*

It is proved that a continuous quasi norm can always be defined on a locally bounded linear topological space. This answers a question of Hyers (Revista de Ciencias vol. 4 (1939) pp. 558–574) in the affirmative. The multiplier of the quasi norm is not uniquely determined by the l.t.s. and it is shown by means of an example that there need be no "best" multiplier. (Received November 24, 1942.)

93. D. E. Christie: *Net homotopy for compacta.*

Let R be a compactum embedded in a parallelotope P. Two mappings f and g of a topological space S into R are said to be neighborhood-homotopic if they are homotopic in every neighborhood of R in P. The notion of neighborhood-homotopy leads to corresponding homotopy groups. These groups are independent of the particular embedding of R in P since they are equivalent to similar groups defined in terms of the net (sense of Lefschetz) of all open coverings of R. This equivalence is demonstrated by means of a study of the Kuratowski mappings of neighborhoods of R into their nerves. The point of view adopted for homotopies is next applied to mappings. Both net and neighborhood definitions are developed: the former lead to easier proofs and wider applications, but the latter are more intuitive. The homotopy groups obtained from these new considerations are significant even for non-arcwise-connected spaces; they are suitably related to Čech homology even for spaces subject to no restrictions about local connectedness; yet they agree with the Hurewicz homotopy groups for absolute neighborhood retracts. (Received November 23, 1942.)

94. H. S. M. Coxeter: *The map-coloring of unorientable surfaces.*

Heawood proved that every map on an unbounded surface of characteristic $K < 2$

can be colored with $[F_K]$ colors, where $F_K=(1/2)(7+(49-24K)^{1/2})$, and he conjectured that this number of colors is necessary as well as sufficient. Franklin disproved this conjecture in the case of the unorientable surface with $K=0$. On the other hand, the necessity of $[F_K]$ colors has been verified in many other cases by considering a map of this number of regions, each touching all the others; for example, Bose discovered a "balanced incomplete block design" which can be interpreted as a map of ten enneagons ($K=-5$). In the present paper this map is shown to be related to two of the stellated icosahedra in ordinary space; and a map of nine octagons ($K=-3$) is derived from a four-dimensional polytope. (Received November 5, 1942.)

95. Samuel Eilenberg and Saunders MacLane: *On a group construction by Hopf.*

In studying the influence of the fundamental group of a polyhedron on its second homology group H. Hopf (Comment. Math. Helv. vol. 14 (1942) pp. 257–309) constructs for each discrete group G an abelian group G_1^* and shows that when G is the fundamental group, G_1^* is a suitable quotient group of the second homology group. Hopf's definition of G_1^* uses a representation of G by means of generators and relations. In order to make the definition intrinsic the authors consider the central extensions of the group P of reals reduced mod 1 by the group G. The equivalence classes of such extensions form a compact abelian group ExtCent $\{P, G\}$ whose character group is shown to be naturally isomorphic with G_1^*. The result is based on a generalization of a previous result of the authors concerning abelian group extensions (see Bull. Amer. Math. Soc. abstract 48-1-94). (Received November 23, 1942.)

96. L. M. Graves: *Metrisation of weak convergence in Banach spaces.*

In the topology of weak convergence in a Banach space, derived sets need not be closed, so that weak convergence cannot in general be expressed in terms of a metric. In this paper a class of cases is characterized in which a simple explicit solution can be given for the problem of the metrisation of weak convergence in certain subsets of a Banach space or of its adjoint. (Received October 29, 1942.)

97. Karl Menger and S. G. Reed: *On a surface not intersecting the set R_3^1.*

By the sum theorem of dimension theory R_3^1, the set of all points of the 3-space with exactly one rational coordinate, is 0-dimensional. As O. Schreier remarked, R_3^1 intersects each surface $z=f(x, y)$ where f is a continuous function defined on a square in the x, y-plane. The boundary of a neighborhood not intersecting R_3^1 can be constructed by topping a cube with steppyramids and iterating this process on each of the cubes of which the step consists. (Received October 28, 1942.)

98. A. N. Milgram: *A topologically invariant metric property of simple closed curves.*

Recently L. M. Blumenthal raised the question as to whether each simple closed curve contains three points which are vertices of an equilateral triangle. This question seems heretofore to have been unanswered even for closed plane curves. In this paper it is shown that: 1. Each simple closed plane curve contains the vertices of a triangle similar to any prescribed triangle. 2. Each closed polygon in euclidean n-space has the same property. (Received October 28, 1942.)

99. C. E. Rickart: *Decomposition of additive set functions.*

An additive set function $x(e)$ defined in a σ-field $\mathfrak{M}$ to a Banach space is said to satisfy property (A) provided $\lim_{n\to\infty} x(e_n)=0$, for any disjoint sequence $\{e_n\}\subset\mathfrak{M}$. If $x(e)$ satisfies (A) and Γ is any σ-ideal contained in $\mathfrak{M}$, then there exists a unique decomposition of $x(e)$ of the form $x(e)=x_1(e)+x_2(e)$, where there exists $e_0\in\Gamma$ such that $x_1(e)=x_1(ee_0)$ and $x_2(e)=0$ for every $e\in\Gamma$. A consequence of this result is that, for an arbitrary outer measure $u(e)$ defined over $\mathfrak{M}$, every completely additive function $x(e)$ can be decomposed uniquely into the form $x(e)=x_1(e)+x_2(e)$, where $x_1(e)=x_1(ee_0)$, $u(e_0)=0$, and $x_2(e)$ is absolutely continuous relative to $u(e)$ (that is, $\lim_{n\to\infty} u(e_n)=0$ implies $\lim_{n\to\infty} x_2(e_n)=0$). (Received November 19, 1942.)

100. R. L. Swain: *Approximate isometries in bounded spaces.*

Approximate isometries have been studied by S. M. Ulam and D. H. Hyers (abstract 47-9-427). The present paper exhibits an example to show that results of the type which they obtained cannot generally be obtained in bounded spaces. Also the following theorem is proved: Let M be a subset of a compact metric space S and let η be a positive number. Then there exists a positive number ϵ such that if T is any ϵ-isometry of M (within S), there is an isometry U of M such that for each point x of M, the distance between $T(x)$ and $U(x)$ is less than η. (Received October 30, 1942.)

101. A. D. Wallace: *On non-alternating transformations.*

The H-spaces M and N are assumed to be compact and connected and Δ will denote a continuous transformation from M onto N. If Δ is non-alternating it is known that (a) for each p-chain Y there exists a unique p-chain X such that $Y\subset\Delta X$; (b) if X is a chain then so also is ΔX. To this may be added, if M is locally connected (c) if A and B are intersecting chains then $\Delta(AB)=(\Delta A)(\Delta B)$. This is no longer true if the hypothesis of local connectivity is deleted. It may be shown (Δ non-alternating) that (d) if $a\sim b$ and a continuum meets both $\Delta^{-1}a$ and $\Delta^{-1}b$ then the continuum contains points a_0, b_0 with $a=\Delta a_0$, $b=\Delta b_0$ such that $a_0\sim b_0$; (e) if K is a continuum and $\Delta^{-1}K=A+B$ is a separation then there are points a_1, b_1 in A and B such that $a_1\sim b_1$. If it is assumed that Δ satisfies (b), (d) and (e) then (a) holds even if Δ alternates. The proof of (c) may be based on the following characterization of a chain: In order that the closed set C be a chain it is n.a.s. that it satisfy both (i) if a, b are distinct points of C and $a\sim x\sim b$ then x is in C and (ii) if p separates the points s and t of C in the space, then p is in C. (Received October 26, 1942.)

102. G. S. Young: *Sets of axioms for the plane.* Preliminary report.

R. L. Moore has given several sets of axioms characterizing the plane, of which the Axioms 0–8 of his *Foundations of point set theory* (Amer. Math. Soc. Colloquium Publications vol. 13, 1932) are probably the most satisfactory. The purpose of this paper is to present several other sets of axioms also defining the plane, which postulate such properties as the two-sidedness of arcs, the non-existence of primitive skew curves, separation by open curves, non-separation by arcs, and so on. Axioms 0–2 of *Foundations* have been assumed throughout. The axiom sets considered divide into two classes: those in which the primary intent has been to state axioms equivalent to Moore's Axioms 3–5; and those in which the purpose was to furnish substitutes for Axioms 5–8. It is thought that these alternative approaches to the plane may prove useful in, for example, imbedding theorems. (Received October 28, 1942.)